The Mathematics of Optimal Execution

Olivier Guéant (Université Paris 1 Panthéon-Sorbonne)
CFM-Imperial Distinguished Lectures

Fall 2016

General introduction

The lecturer

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- Current position: Full Professor of Applied Mathematics at Univ. Paris 1 Panthéon Sorbonne.

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- Past: Professor of Quantitative Finance at ENSAE, Assistant Professor at Univ. Paris 7.
- Research: initially in mean field games (PhD), then in Quantitative Finance:
  - Optimal execution,
  - Market making,
  - Option pricing,
  - Asset management.

No quarrel between the Ancients and the Moderns

New research strands

- Before 2007, derivatives pricing held the lion's share in the academic/professional research landscape.

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  2. new regulations and changes following MiFID (Europe) and Reg. NMS (US),
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Main goals of the lectures

- Presenting classical models/approaches for optimal execution.

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- Presenting classical models/approaches for optimal execution.
- Showing that these models/approaches can be used to address classical problems in a different way.

A set of three lectures

Today: The Almgren-Chriss model revisited

- The Almgren-Chriss model and some generalizations.
- How it can be used in the cash-equity/brokerage industry.

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- Vanilla option pricing and hedging.
- Accelerated Share Repurchase (ASR) contracts.

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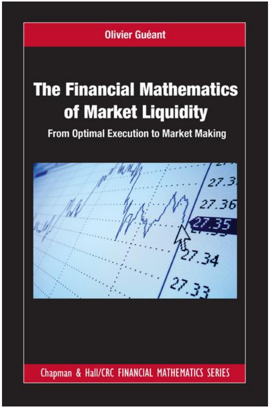
Next week: Asset management with execution costs

- Markowitz/Merton in the Almgren-Chriss framework.
- Introduction of Bayesian learning.

The book: self-advertizing



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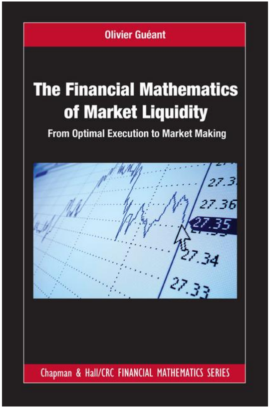

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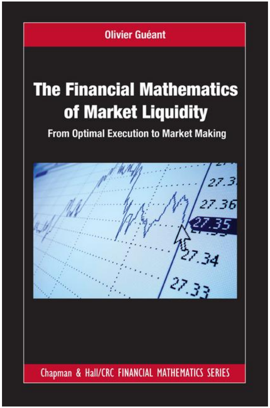

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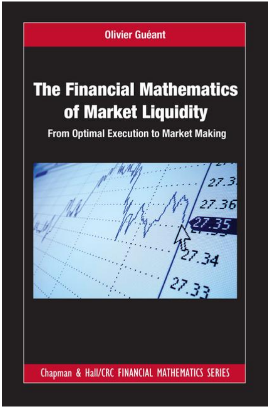

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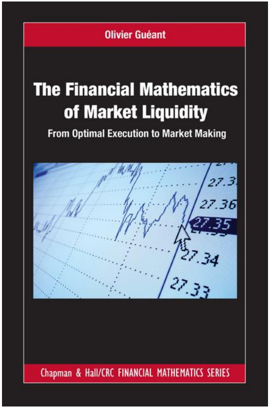

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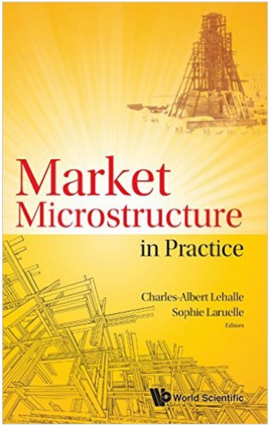
- Asset management (third lecture) is not covered.
- The book also addresses the history of stock exchanges and the mathematics of market making.

Other interesting books



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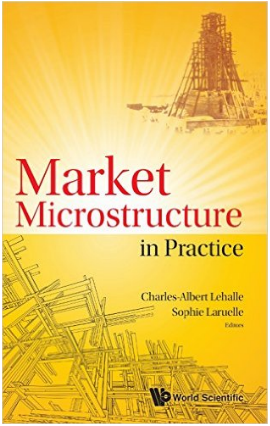

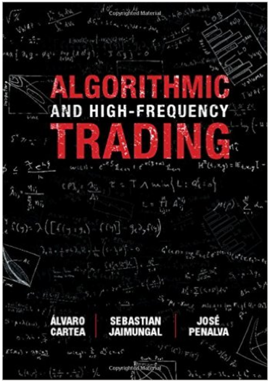

Lecture 1:
The Almgren-Chriss model revisited.

Introduction

Optimal Liquidation

Basic question:
How to optimally liquidate a portfolio with q_0 shares?

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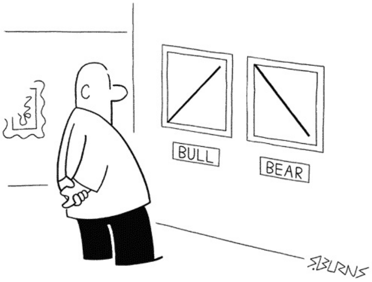
Classical trade-off

- Liquidating fast is costly: execution costs and market impact.
- But if one liquidates too slowly...



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Need to find an optimal trading schedule.

The original Almgren-Chriss model

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- Time: $t_0 = 0 < \ldots < t_n = n\Delta t < \ldots < t_N = N\Delta t = T$.

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- Price: $S_{n+1} = S_n + \sigma\sqrt{\Delta t}\epsilon_{n+1} - kv_{n+1}\Delta t$.

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- Price: $S_{n+1} = S_n + \sigma\sqrt{\Delta t}\epsilon_{n+1} - kv_{n+1}\Delta t$.
- Cash: $X_{n+1} = X_n + v_{n+1}S_n\Delta t - \eta v_{n+1}^2\Delta t$.

The random variables $(\epsilon_n)_n$ are i.i.d. $\mathcal{N}(0,1)$ variables.

The original Almgren-Chriss framework

Optimization problem

Maximizing

$$\mathbb{E}\left[X_N\right] - \frac{\gamma}{2}\mathbb{V}\left[X_N\right].$$

over

$$(v_n)_n \in \mathcal{A}_d = \left\{(v_1, \ldots, v_N) \in \mathbb{R}^N, \sum_{n=0}^{N-1} v_{n+1}\Delta t = q_0\right\}.$$

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Cash account at time $t_N = T$

$$\begin{aligned} X_N &= X_0 + q_0 S_0 - \frac{k}{2}q_0^2 + \sigma\sqrt{\Delta t}\sum_{n=0}^{N-1} q_{n+1}\epsilon_{n+1} \\ &\quad - \sum_{n=0}^{N-1} \underbrace{\left(\eta - \frac{k}{2}\Delta t\right)}_{=\tilde{\eta}>0} v_{n+1}^2 \Delta t. \end{aligned}$$

The original Almgren-Chriss framework

Moments of X_N

$$\mathbb{E}\left[X_N\right] = X_0 + q_0 S_0 - \frac{k}{2} q_0^2 - \sum_{n=0}^{N-1} \tilde{\eta} v_{n+1}^2 \Delta t.$$

$$\mathbb{V}\left[X_N\right] = \sigma^2 \Delta t \sum_{n=0}^{N-1} q_{n+1}^2.$$

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Minimization problem

$$\sum_{n=0}^{N-1} \tilde{\eta} v_{n+1}^2 \Delta t + \frac{\gamma}{2} \sigma^2 \Delta t \sum_{n=0}^{N-1} q_{n+1}^2$$

$$= \sum_{n=0}^{N-1} \frac{\tilde{\eta}}{\Delta t} (q_n - q_{n+1})^2 + \frac{\gamma}{2} \sigma^2 \Delta t \sum_{n=0}^{N-1} q_{n+1}^2.$$

The original Almgren-Chriss framework

First order condition

The minimizer q^* is the solution of the second-order recursive equation

$$q^*_{n+2} - \left(2 + \frac{\gamma\sigma^2}{2\tilde{\eta}}\Delta t^2\right) q^*_{n+1} + q^*_n = 0,$$

with boundary conditions $q^*_0 = q_0$ and $q^*_N = 0$.

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Solution: the sinh formula (in discrete time)

$$q^*_n = q_0 \frac{\sinh\left(\alpha(T - t_n)\right)}{\sinh\left(\alpha T\right)},$$

where α is the unique positive solution of

$$2\left(\cosh(\alpha\Delta t) - 1\right) = \frac{\gamma\sigma^2}{2\tilde{\eta}}\Delta t^2.$$

A generalized version of the Almgren-Chriss model (in continuous time)

Almgren-Chriss model: a general framework

We consider the liquidation of $q_0 > 0$ shares.

Framework in continuous time with 4 variables

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where $(V_t)_t$ is the market volume curve, assumed to be deterministic.

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L is strictly convex, (even), asymptotically superlinear, increasing on $\mathbb{R}_+$, with $L(0) = 0$. In practice:

$$L(\rho) = \eta|\rho|^{1+\phi} + \psi|\rho|$$

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Optimization problem

$$\sup_{(v_t)_t \in \mathcal{A}} \mathbb{E}\left[-\exp(-\gamma X_T)\right]$$

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Admissible strategies are related to Implementation Shortfall (IS) orders with/without participation constraints:

$$\mathcal{A}_{without} = \left\{ (v_t)_{t\in[0,T]} \text{ prog mes }, \int_0^T |v_t| dt \in L^\infty, \int_0^T v_t dt = q_0 \right\}$$

$$\mathcal{A}_{with} = \left\{ (v_t)_{t\in[0,T]} \text{ prog mes }, |v_t| \leq \rho_{\max} V_t, \int_0^T v_t dt = q_0 \right\}$$

Almgren-Chriss model: a general framework

Expression of X_T

$$X_T = X_0 + q_0 S_0 - \frac{k}{2} q_0^2 + \sigma \int_0^T q_t dW_t - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt.$$

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Remark: the cost associated with permanent market impact is independent of the strategy.

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Law of X_T

If $v \in \mathcal{A}$ is deterministic, then X_T is normally distributed with:

- mean: $q_0 S_0 - \frac{k}{2} q_0^2 - \int_0^T V_s L\left(\frac{v_s}{V_s}\right) ds$
- variance: $\sigma^2 \int_0^T q_s^2 ds$.

Almgren-Chriss model: a general framework

By taking the Laplace transform, the problem boils down to the following minimization problem:

Minimization problem

$$\inf_{q \in W^{1,1}_{q_0,0}(0,T)} \mathcal{I}(q),$$

where

$$\mathcal{I}(q) = \int_0^T \left(V_s L\left(\frac{\dot{q}(s)}{V_s}\right) + \frac{1}{2}\gamma\sigma^2 q^2(s) \right) ds.$$

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Theorem (Existence and uniqueness of a minimizer)

There exists a unique minimizer $q \in W^{1,1}_{q_0,0}(0,T)$ of $\mathcal{I}$. This minimizer is a nonnegative and nonincreasing function.

Almgren-Chriss model: a general framework

Hamiltonian characterization

$$\begin{cases} \dot{p}(t) &= \gamma\sigma^2 q(t) \\ \dot{q}(t) &= V_t H'(p(t)) \end{cases} \qquad q(0) = q_0, \quad q(T) = 0,$$

where $H(p) = \sup_{|\rho| \leq \rho_{\max}} \rho p - L(\rho)$ or $H(p) = \sup_{\rho} \rho p - L(\rho)$

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Quadratic case and flat volume curve: a linear ODE

If $L(\rho) = \eta\rho^2$ and $V_t = V$ then $H(p) = \frac{p^2}{4\eta}$, and

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$$\Rightarrow q(t) = q_0 \frac{\sinh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}(T-t)\right)}{\sinh\left(\sqrt{\frac{\gamma\sigma^2 V}{2\eta}}T\right)}.$$

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$$\times\mathbb{E}\left[\exp\left(\gamma\int_0^T V_t L\left(\frac{v_t}{V_t}\right)dt\right)\exp\left(-\gamma\sigma\int_0^T q_t dW_t\right)\right]$$

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where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\gamma\sigma\int_0^T q_t dW_t - \frac{1}{2}\gamma^2\sigma^2\int_0^T q_t^2 dt\right).$$

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$$\leq -\exp\left(-\gamma\left(X_0 + q_0 S_0 - \frac{k}{2}q_0^2\right)\right)\exp\left(\gamma\mathcal{I}(q^*)\right),$$

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and equality is obtained for the deterministic strategy q^*.

This result means that there is an optimal trading curve, computable *ex-ante*.

Numerical methods and examples

Discretization of the Hamiltonian system

Hamiltonian equations

$$\begin{cases} p'(t) & = & \gamma\sigma^2 q^*(t), \\ {q^*}'(t) & = & V_t H'(p(t)), \\ q^*(0) & = & q_0, \\ q^*(T) & = & 0, \end{cases}$$

Discretization of the Hamiltonian system

Hamiltonian equations

$$\begin{cases} p'(t) &=& \gamma\sigma^2 q^*(t), \\ {q^*}'(t) &=& V_t H'(p(t)), \\ q^*(0) &=& q_0, \\ q^*(T) &=& 0, \end{cases}$$

Discrete-time equivalent

$$\begin{cases} p_{n+1} &=& p_n + \Delta t \gamma\sigma^2 q^*_{n+1}, \quad 0 \le n < N-1, \\ q^*_{n+1} &=& q^*_n + \Delta t V_{n+1} H'(p_n), \quad 0 \le n < N, \\ q^*_0 &=& q_0, \\ q^*_N &=& 0. \end{cases}$$

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We face a problem with initial and final conditions. It requires a fixed-point approach.

Numerical methods

Shooting method (simple portfolios)

$$\left\{\begin{array}{rcl} p^{\lambda}_{n+1} & = & p^{\lambda}_{n} + \Delta t \gamma \sigma^2 q^{\lambda}_{n+1}, \quad 0 \le n < N-1, \\ q^{\lambda}_{n+1} & = & q^{\lambda}_{n} + \Delta t V_{n+1} H'(p^{\lambda}_{n}), \quad 0 \le n < N, \\ q^{\lambda}_{0} & = & q_0, \\ p^{\lambda}_{0} & = & \lambda. \end{array}\right.$$

Numerical methods

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$\rightarrow$ Then we need to find λ such that $q_T^{\lambda} = 0$ (by bisection method for instance).

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$\rightarrow$ Then we need to find λ such that $q_T^{\lambda} = 0$ (by bisection method for instance).

Other methods

- Newton's method on the Hamiltonian system.
- Gradient descent on the convex problem.

Examples

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- $S_0 = 45$ €,

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- $\sigma = 0.6$ €$\cdot$day$^{-1/2}\cdot$share^{-1}, i.e., $\simeq 21\%$,

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Examples

- $S_0 = 45$ €,
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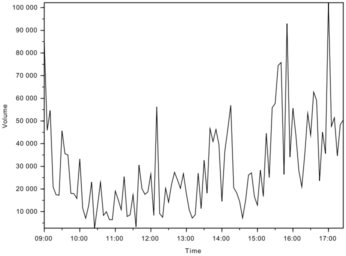
  $\eta = 0.1$ €$\cdot$share^{-1}, $\psi = 0.004$ €$\cdot$share^{-1},
  and $\phi = 0.75$.
- For $(V_t)_t$: average market volume curve over a month.



Examples

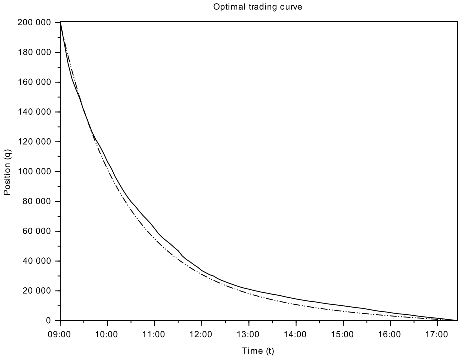


Figure: Optimal trading curve for $q_0 = 200{,}000$ shares over one day ($T = 1$), for different market volume curves. Solid line: market volume curve $(V_t)_t$. Dash-dotted line: flat market volume curve with 4,000,000 shares per day – $\gamma = 5.10^{-6}$ €$^{-1}$, $\rho_{\max} = 5$, so that the constraint is never binding.

Examples

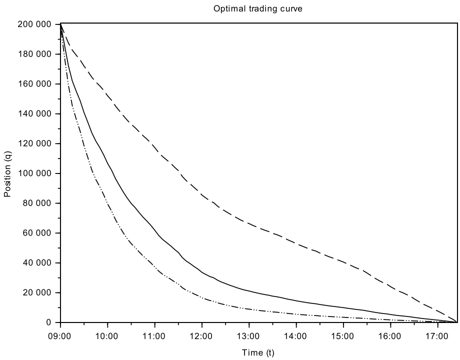


Figure: Optimal trading curve for $q_0 = 200{,}000$ shares over one day ($T = 1$), for different values of γ. Dash-dotted line: $\gamma = 10^{-5}$ €$^{-1}$. Solid line: $\gamma = 5.10^{-6}$ €$^{-1}$. Dashed line: $\gamma = 10^{-6}$ €$^{-1}$ – $\rho_{\max} = 5$, as above.

Examples

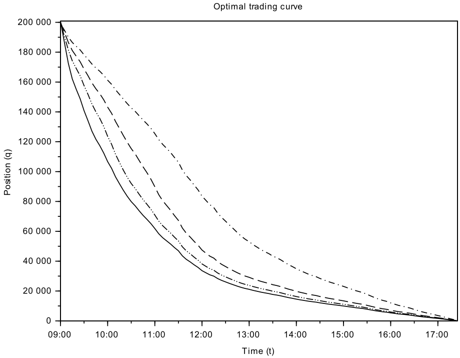


Figure: Optimal trading curve for $q_0 = 200{,}000$ shares over one day ($T = 1$), for different values of $\rho_{\max}$. Solid line: $\rho_{\max} = 5$ (a very high value, such that the constraint is never binding). Dash-dotted line (two dots): $\rho_{\max} = 20\%$. Dashed line: $\rho_{\max} = 15\%$. Dash-dotted line (one dot): $\rho_{\max} = 10\%$.

Multidimensional extensions

Almgren-Chriss model for a multi-asset portfolio

We consider the liquidation of a portfolio with d different assets.

Framework in continuous time with 4 variables

Almgren-Chriss model for a multi-asset portfolio

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Framework in continuous time with 4 variables

- Time: t.

Almgren-Chriss model for a multi-asset portfolio

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Framework in continuous time with 4 variables

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- Number of shares: $q^i_t = q^i_0 - \int_0^t v^i_s ds$.

Almgren-Chriss model for a multi-asset portfolio

We consider the liquidation of a portfolio with d different assets.

Framework in continuous time with 4 variables

- Time: t.
- Number of shares: $q_t^i = q_0^i - \int_0^t v_s^i ds$.
- Price: $dS_t^i = \sigma^i dW_t^i - k^i v_t^i dt$.
  $(\sigma^1 W_t^1, \ldots, \sigma^d W_t^d)_t$ has a nonsingular covariance matrix Σ.

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  $(\sigma^1 W_t^1, \ldots, \sigma^d W_t^d)_t$ has a nonsingular covariance matrix Σ.
- Cash: $dX_t = \sum_{i=1}^d v_t^i S_t^i dt - V_t^i L^i \left(\frac{v_t^i}{V_t^i}\right) dt$.

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- Cash: $dX_t = \sum_{i=1}^d v_t^i S_t^i dt - V_t^i L^i \left(\frac{v_t^i}{V_t^i}\right) dt$.

Remark: no "cross" impact, but interactions between assets through Σ.

Almgren-Chriss model for a multi-asset portfolio

Value of X_T for liquidation strategies

$$X_T = X_0 + \sum_{i=1}^{d} q_0^i S_0^i - \sum_{i=1}^{d} \frac{k^i}{2} {q_0^i}^2$$

$$+ \sum_{i=1}^{d} \int_0^T q_t^i \sigma^i dW_t^i - \sum_{i=1}^{d} \int_0^T V_t^i L^i \left(\frac{v_t^i}{V_t^i} \right) dt.$$

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Optimization problem

$$\sup_{(v_t)_t \in \mathcal{A}} \mathbb{E}\left[-\exp(-\gamma X_T) \right]$$

Almgren-Chriss model for a multi-asset portfolio

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Optimization problem

$$\sup_{(v_t)_t \in \mathcal{A}} \mathbb{E}\left[-\exp(-\gamma X_T) \right]$$

Remark: as in the single-asset case, deterministic strategies are optimal.

Almgren-Chriss model for a multi-asset portfolio

Minimization problem

Minimize

$$J(q) = \int_0^T \left(\sum_{i=1}^d V_t^i L^i \left(\frac{q^{i\prime}(t)}{V_t^i} \right) + \frac{\gamma}{2} q(t) \cdot \Sigma q(t) \right) dt,$$

over the set of $\mathbb{R}^d$-valued absolutely continuous functions $q \in W^{1,1}(0, T)$ satisfying the constraints $q(0) = q_0$ and $q(T) = 0$.

Almgren-Chriss model for a multi-asset portfolio

Minimization problem

Minimize

$$J(q) = \int_0^T \left(\sum_{i=1}^{d} V_t^i L^i \left(\frac{q^{i'}(t)}{V_t^i} \right) + \frac{\gamma}{2} q(t) \cdot \Sigma q(t) \right) dt,$$

over the set of $\mathbb{R}^d$-valued absolutely continuous functions $q \in W^{1,1}(0,T)$ satisfying the constraints $q(0) = q_0$ and $q(T) = 0$.

Hamilton characterization

$$\begin{cases} p'(t) & = & \gamma \Sigma q^*(t), \\ q^{i*'}(t) & = & V_t^i H^{i'}(p^i(t)), \forall i, \\ q^*(0) & = & q_0, \\ q^*(T) & = & 0, \end{cases}$$

with $H^i(p) = \sup_\rho \rho p - L^i(\rho)$.

Examples

Asset 1:

- $S_0 = 100$ €,
- $\sigma = 1.2$ €·day$^{-1/2}$·share^{-1},
- $V = 3{,}000{,}000$ shares·day^{-1},
- $L(\rho) = \eta|\rho|^{1+\phi} + \psi|\rho|$, where $\eta = 0.5$ €·share^{-1}, $\phi = 0.5$, and $\psi = 0.01$ €·share^{-1}.

Examples

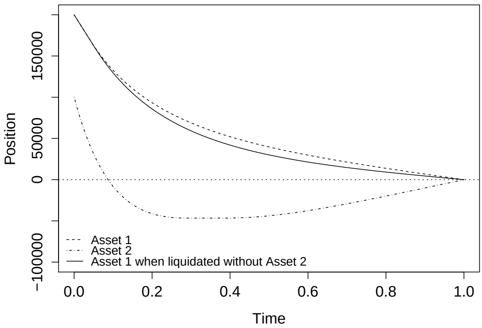
Asset 1:

- $S_0 = 100$ €,
- $\sigma = 1.2$ €·day$^{-1/2}$·share^{-1},
- $V = 3{,}000{,}000$ shares·day^{-1},
- $L(\rho) = \eta|\rho|^{1+\phi} + \psi|\rho|$, where $\eta = 0.5$ €·share^{-1}, $\phi = 0.5$, and $\psi = 0.01$ €·share^{-1}.

Asset 2:

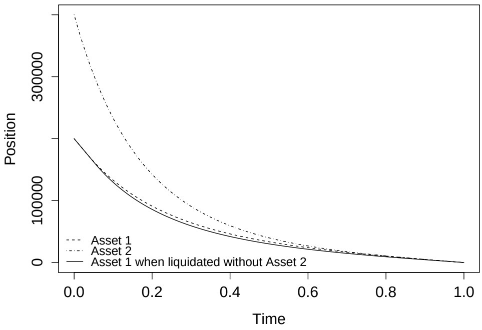
- $S_0 = 45$ €,
- $\sigma = 0.6$ €·day$^{-1/2}$·share^{-1}
- $V = 4{,}000{,}000$ shares·day^{-1},
- $L(\rho) = \eta|\rho|^{1+\phi} + \psi|\rho|$, where $\eta = 0.1$ €·share^{-1}, $\phi = 0.75$, and $\psi = 0.004$ €·share^{-1}.

Examples



Optimal trading curves for a two-stock portfolio – correlation 80%, $\gamma = 5.10^{-6}$ €$^{-1}$.

Examples



Optimal trading curves for a two-stock portfolio – correlation -20%, $\gamma = 5.10^{-6}$ €$^{-1}$.

Target Close orders

Almgren-Chriss for Target Close orders

Target Close orders

- Many agents want their orders to be executed at a price close to the closing price of the day.

Almgren-Chriss for Target Close orders

Target Close orders

- Many agents want their orders to be executed at a price close to the closing price of the day.
  - Closing auction: possible for not too large orders, algorithms for large orders.

Almgren-Chriss for Target Close orders

Target Close orders

- Many agents want their orders to be executed at a price close to the closing price of the day.
  - Closing auction: possible for not too large orders, algorithms for large orders.
  - No closing auction: an algorithm is needed.

Almgren-Chriss for Target Close orders

Target Close orders

- Many agents want their orders to be executed at a price close to the closing price of the day.
  - Closing auction: possible for not too large orders, algorithms for large orders.
  - No closing auction: an algorithm is needed.
- Fixed quantity to trade at the closing auction (if any). The remainder traded during the continuous auction.

Almgren-Chriss for Target Close orders

Target Close orders

- Many agents want their orders to be executed at a price close to the closing price of the day.
  - Closing auction: possible for not too large orders, algorithms for large orders.
  - No closing auction: an algorithm is needed.
- Fixed quantity to trade at the closing auction (if any). The remainder traded during the continuous auction.

Dynamics during the continuous auction

- Number of shares: $q_t = q_0 - \int_0^t v_s ds$.
- Price: $dS_t = \sigma dW_t$.
- Cash: $dX_t = v_t S_t dt - V_t L\left(\frac{v_t}{V_t}\right) dt$.

Almgren-Chriss for Target Close orders

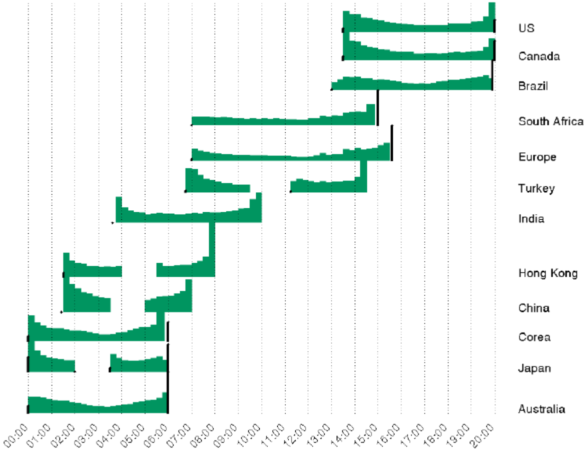


Figure 2.5. Intraday volume patterns across the globe.

Intraday volume curves and auctions (credit: C.-A. Lehalle).

Almgren-Chriss for Target Close orders

Auction (v_{close} fixed *ex-ante*)

$$S_{\mathrm{close}} = S_T + \sigma_{\mathrm{close}}\epsilon,$$

$$X_{\mathrm{close}} = X_T + v_{\mathrm{close}} S_{\mathrm{close}}.$$

Almgren-Chriss for Target Close orders

Auction (v_{close} fixed *ex-ante*)

$$S_{\text{close}} = S_T + \sigma_{\text{close}}\epsilon,$$

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$$\mathcal{A} = \left\{ (v_t)_{t\in[0,T]}, \int_0^T |v_t| dt \in L^\infty, \int_0^T v_t dt + v_{\text{close}} = q_0 \right\}$$

Almgren-Chriss for Target Close orders

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Optimization problem

$$\sup_{(v_t)_t \in \mathcal{A}} \mathbb{E}\left[-\exp(-\gamma(X_{\text{close}} - X_0 - q_0 S_{\text{close}})) \right]$$

Almgren-Chriss for Target Close orders

$$X_{\mathrm{close}} - X_0 - q_0 S_{\mathrm{close}} =$$
$$-(q_0 - v_{\mathrm{close}})\sigma_{\mathrm{close}}\epsilon - \sigma \int_0^T (q_0 - q_t) dW_t - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt.$$

Almgren-Chriss for Target Close orders

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Minimization problem

$$\inf_{q \in W^{1,1}_{q_0, v_{\text{close}}}(0,T)} \mathcal{I}_{\text{close}}(q),$$

where

$$\mathcal{I}_{\text{close}}(q) = \int_0^T \left(V_s L\left(\frac{\dot{q}(s)}{V_s}\right) + \frac{1}{2}\gamma\sigma^2 (q_0 - q(s))^2 \right) ds.$$

Almgren-Chriss for Target Close orders

$$\tilde{q}(t) = q_0 - q(T - t), \qquad \tilde{V}_t = V_{T-t}.$$

Almgren-Chriss for Target Close orders

$$\tilde{q}(t) = q_0 - q(T-t), \qquad \tilde{V}_t = V_{T-t}.$$

New minimization problem

$$\inf_{\tilde{q} \in W^{1,1}_{q_0 - v_{\text{close}},0}(0,T)} \tilde{J}(\tilde{q}),$$

where

$$\tilde{J}(\tilde{q}) = \int_0^T \left(\tilde{V}_t L\left(\frac{\tilde{q}'(t)}{\tilde{V}_t}\right) + \frac{1}{2}\gamma\sigma^2 \tilde{q}(t)^2 \right) dt.$$

Same problem as for an IS order with $q_0 - v_{\text{close}}$ shares (with time-reversed volume curve).

Example

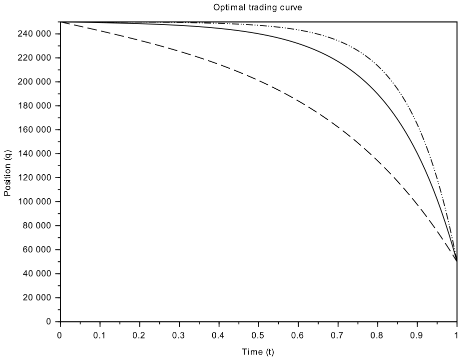

Figure: Optimal trading curves for a Target Close order for $q_0 = 250{,}000$ shares over one day ($T = 1$), when $v_{\text{close}} = 50{,}000$ shares, for different values of γ. Dash-dotted line: $\gamma = 10^{-5}$ €$^{-1}$. Solid line: $\gamma = 5.10^{-6}$ €$^{-1}$. Dashed line: $\gamma = 10^{-6}$ €$^{-1}$.

POV orders

POV orders

Different kinds of orders

- Implementation Shortfall orders (classical AC).
- Target close orders (reverse IS).
- POV orders (with defined participation rate).
- VWAP orders (see Konishi, McCulloch and Kazakov, Frei and Westray, etc.).
- etc.

POV orders

Different kinds of orders

- Implementation Shortfall orders (classical AC).
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- POV orders (with defined participation rate).
- VWAP orders (see Konishi, McCulloch and Kazakov, Frei and Westray, etc.).
- etc.

Goals

- Determine the optimal rate for POV orders as a function of the parameters.
- Find a way to choose the risk aversion parameter γ.

POV orders

Optimization problem

$$\sup_{(v_t)_t \in \mathcal{A}_{POV}} \mathbb{E}\left[-\exp(-\gamma X_T)\right],$$

where T is a time such that $\int_0^T v_t dt = q_0$.

- T is not fixed *ex-ante*.
- The set of admissible strategies is

$$\mathcal{A}_{POV} = \left\{ (v_t)_t, \exists \rho \in \mathbb{R}_+^*, v_t = \rho V_t 1_{\int_0^t v_s ds \leq q_0} \right\}.$$

POV orders

Cash account at time T

$$X_T = q_0 S_0 - \frac{k}{2} q_0^2 - \frac{L(\rho)}{\rho} q_0 + \sigma\rho \int_0^T \int_t^T V_s \, ds \, dW_t.$$

POV orders

Cash account at time T

$$X_T = q_0 S_0 - \frac{k}{2} q_0^2 - \frac{L(\rho)}{\rho} q_0 + \sigma \rho \int_0^T \int_t^T V_s ds dW_t.$$

If we take the Laplace transform, the problem boils down to minimizing

Expression to minimize

$$\frac{L(\rho)}{\rho} q_0 + \frac{\gamma}{2} \sigma^2 \rho^2 \int_0^T \left(\int_t^T V_s ds \right)^2 dt$$

POV orders

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Expression to minimize

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If the volume curve is flat ($V_s = V$), then:

$$\frac{L(\rho)}{\rho} q_0 + \frac{\gamma}{6} \sigma^2 \frac{q_0^3}{\rho V}$$

POV orders

Optimal participation rate if $L(\rho) = \eta\rho^{1+\phi} + \psi|\rho|$

$$\rho^* = \left(\frac{\gamma\sigma^2}{6\eta\phi}\frac{q_0^2}{V}\right)^{\frac{1}{1+\phi}}.$$

POV orders

Optimal participation rate if $L(\rho) = \eta\rho^{1+\phi} + \psi|\rho|$

$$\rho^* = \left(\frac{\gamma\sigma^2}{6\eta\phi}\frac{q_0^2}{V}\right)^{\frac{1}{1+\phi}}.$$

- Does not depend on permanent market impact.
- Does not depend on ψ.
- Increasing with γ (risk aversion), σ (volatility), q_0 (inventory)
- Decreasing with η (illiquidity), ϕ (when $\rho \leq 1$)
- $\rho^* V$ (volume we trade per unit of time) is increasing in V (average daily volume).

Choice of γ

Inversion of the formula

$$\rho^* = \left(\frac{\gamma \sigma^2}{6\eta\phi} \frac{q_0^2}{V} \right)^{\frac{1}{1+\phi}}$$

$$\Rightarrow \gamma = \frac{6\eta\phi V {\rho^*}^{1+\phi}}{\sigma^2 q_0^2}.$$

Choice of γ

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- The above formula is a way to discover/reveal one's risk aversion.

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- An empirical study could be carried out on cash equity desks.

Choice of γ

Inversion of the formula

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$$\Rightarrow \gamma = \frac{6\eta\phi V {\rho^*}^{1+\phi}}{\sigma^2 q_0^2}.$$

- γ has to be chosen.
- The above formula is a way to discover/reveal one's risk aversion.
- An empirical study could be carried out on cash equity desks.

Remark: We will discuss in Lecture 2 another way to choose γ.

Final remarks

Strategies and tactics

A two-step process

The two-step approach is legitimated by the optimality of deterministic strategies (in the model):

Strategies and tactics

A two-step process

The two-step approach is legitimated by the optimality of deterministic strategies (in the model):

- Step 1 (strategies): Optimal scheduling – trading curve (the problem addressed by Almgren and Chriss).

Strategies and tactics

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The two-step approach is legitimated by the optimality of deterministic strategies (in the model):

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- Step 2 (tactics): Optimal tactics to follow the trading curve.

Strategies and tactics

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- Step 1 (strategies): Optimal scheduling – trading curve (the problem addressed by Almgren and Chriss).
- Step 2 (tactics): Optimal tactics to follow the trading curve.

Tactics

- Decomposition into slices.

Strategies and tactics

A two-step process

The two-step approach is legitimated by the optimality of deterministic strategies (in the model):

- Step 1 (strategies): Optimal scheduling – trading curve (the problem addressed by Almgren and Chriss).
- Step 2 (tactics): Optimal tactics to follow the trading curve.

Tactics

- Decomposition into slices.
- Child order placement (venue, limit/marketable limit order, price, timing, etc.).

Strategies and tactics

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- Child order placement (venue, limit/marketable limit order, price, timing, etc.).

$\rightarrow$ Many heuristical methods.
$\rightarrow$ Several interesting approaches: Cont-Kukanov, Guilbaud-Pham, reinforcement learning, etc.

Adaptive strategies?

The limits of the two-step process

Adaptive strategies are needed for taking account of:

- changes in volume expectation (intraday or at the close),
- changes in market impact,
- changes in market trend.

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- changes in market trend.

Possible to mix learning and optimal control (see Lecture 3 for an instance).
Often forced to use heuristic methods.

Market impact estimation – a very diversified literature

Several notions

- Single-order impact.
- Price return and volume imbalance (market data).
- Metaorder impact.

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- Market data.
- Exchange data.
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Several data sources

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Different approaches

- Empirical approaches.
- Theoretical approaches (see works by people from CFM to reconcile random walks for prices and the long-range autocorrelation of the order flow).

Market impact estimation

Model à la Almgren-Chriss

- Estimation by Almgren and coauthors from Citigroup on (Citigroup) execution data.
- Many in-house estimations in brokerage companies / on cash-equity desks.

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Transient market impact

In fact market impact is transient:

- Dynamic increase of the price.
- Square-root law.
- Decay.
- Permanent market impact vs. α.

Market impact estimation

Many interesting papers

- Moro *et al.* (Spanish Stock Market and LSE)
- Tóth *et al.* (CFM data – on futures)
- Brokmann *et al.* (CFM data)
- Bershova and Rakhlin (AllianceBernstein data)
- Bacry *et al.* (Cheuvreux data)

End of Lecture 1

Thank you. Questions?

Lecture 2: Pricing in the Almgren-Chriss framework.

Introduction

Main questions

From optimization to pricing

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- Lecture 1: how to liquidate a portfolio of q_0 shares?

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The pricing and hedging of derivatives

The Almgren-Chriss model can be used outside of the cash-equity world.

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The Almgren-Chriss model can be used outside of the cash-equity world.

- What happens to the pricing and hedging of derivatives when one takes account of market impact/execution costs.
- How can we generalize classical results for vanilla options?
- How can we use the Almgren-Chriss model to price and hedge ASR contracts?

Block trade pricing

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Pricing approach

- Indifference pricing: the maximum price that one can pay to get the shares and liquidate them (with nonnegative expected utility).
- This price takes account of:
  - market impact / execution costs,
  - price risk.

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Question: what should be the price for a block of $q_0 > 0$ shares?

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Indifference price $P(T, q_0, S_0)$

$$\sup_{(v_t)_t \in \mathcal{A}} \mathbb{E}\left[-\exp(-\gamma(X_T - X_0))\right] = -\exp(-\gamma P(T, q_0, S_0)),$$

with or without constraints.

The value function $\theta_T(t, q)$

Link with the value function

Using the results of Lecture 1 on IS orders, we find:

$$P(T, q_0, S_0) = q_0 S_0 - \frac{k}{2} q_0^2 - \theta_T(0, q_0),$$

The value function $\theta_T(t, q)$

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where θ_T is the value function:

$$\theta_T(t, q) = \inf_{\tilde{q} \in W^{1,1}_{q,0}(t,T)} \int_t^T \left(V_s L\left(\frac{\tilde{q}'(s)}{V_s}\right) + \frac{1}{2}\gamma\sigma^2 \tilde{q}^2(s) \right) ds.$$

The value function $\theta_T(t, q)$ and the HJ equation

Proposition (Hamilton-Jacobi equation)

θ_T is a locally Lipschitz viscosity solution of the Hamilton-Jacobi equation:

$$-\partial_t \theta_T(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + V_t H(\partial_q \theta_T(t,q)) = 0, \qquad \text{on } [0,T) \times \mathbb{R}.$$

with

$$\lim_{t \to T} \theta_T(t,q) = \begin{cases} 0, & \text{if } q = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

The value function $\theta_T(t, q)$ and the first BTP formula

Proposition (Asymptotic behavior)

In the flat volume curve $V_t = V$ case, if H is increasing on $\mathbb{R}_+$, then:

$$\lim_{T \to +\infty} \theta_T(t, q) = \theta_\infty(q) = \int_0^q H^{-1}\left(\frac{\gamma\sigma^2}{2V} x^2\right) dx,$$

where H^{-1} is the inverse of $H : \mathbb{R}_+ \to \mathbb{R}_+$.

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Block trade pricing formula I

$$P(q, S) = qS - \frac{k}{2} q^2 - \int_0^q H^{-1}\left(\frac{\gamma\sigma^2}{2V} x^2\right) dx$$

We call $qS - P(q, S)$ a risk-liquidity premium/discount.

Block trade pricing formula

If $L(\rho) = \eta|\rho|^{1+\phi} + \psi|\rho|$ and without participation constraints:

$$P(q, S) = qS - \ell(q)$$

where

$$\ell(q) = \frac{k}{2}q^2 + \psi q + \frac{\eta^{\frac{1}{1+\phi}}}{\phi^{\frac{\phi}{1+\phi}}} \frac{(1+\phi)^2}{1+3\phi} \left(\frac{\gamma\sigma^2}{2V}\right)^{\frac{\phi}{1+\phi}} q^{\frac{1+3\phi}{1+\phi}}$$

is the risk-liquidity discount/premium in this particular case.

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is the risk-liquidity discount/premium in this particular case.

This type of premium/discount gives a price to liquidity: it can be used in many problems as a penalization function (and to choose γ).

POV Block trade pricing

In the case of POV orders, we can consider the certainty equivalent and we obtain:

$P(q_0)$ and liquidity premium

$$P(q_0) = \underbrace{q_0 S_0}_{\text{MtM value}} - \underbrace{\frac{k}{2} q_0^2}_{\text{perm. m.i.}}$$

$$\underbrace{-\psi q_0 - \eta^{\frac{1}{1+\phi}} \left(\frac{\gamma\sigma^2}{6\phi V}\right)^{\frac{\phi}{1+\phi}} q_0^{\frac{1+3\phi}{1+\phi}}}_{\text{exec. costs}}$$

$$\underbrace{-\phi\eta^{\frac{1}{1+\phi}} \left(\frac{\gamma\sigma^2}{6\phi V}\right)^{\frac{\phi}{1+\phi}} q_0^{\frac{1+3\phi}{1+\phi}}}_{\text{price risk}}.$$

Comparison between IS and POV

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An interesting result is:

$$1 \geq \frac{premium_{IS}}{premium_{POV}} \geq 3^{\frac{\phi}{1+\phi}} \frac{1+\phi}{1+3\phi} \geq \frac{e \log(3)}{2\sqrt{3}} \simeq 0.86$$

At most 15% difference between IS and POV in terms of certainty equivalent.

Other questions linked to liquidation and block trade pricing

Other problems can be addressed with the Almgren-Chriss modelling framework:

- VWAP orders,
- Guaranteed VWAP contracts,
- Target Close orders,
- Guaranteed Close contracts,
- etc.

Other questions linked to liquidation and block trade pricing

Other problems can be addressed with the Almgren-Chriss modelling framework:

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- Guaranteed VWAP contracts,
- Target Close orders,
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- etc.

But also problems outside of cash trading...

Vanilla option pricing and hedging

Introduction - Option pricing / hedging

- Classical framework for option pricing: Black-Scholes and extensions $\rightarrow$ frictionless market, price-taker agent
- Sometimes super-replication + transaction costs but...

Introduction - Option pricing / hedging

- Classical framework for option pricing: Black-Scholes and extensions $\rightarrow$ frictionless market, price-taker agent
- Sometimes super-replication $+$ transaction costs but...

Issues

- Not suited for options on illiquid assets.
- Not suited to large-nominal options.
- Not suited when Γ is too large.
- No difference between physical and cash settlement.

Optimal execution and options

Other routes

- Transaction costs (fixed or proportional),
- Supply curve approach (Çetin-Jarrow-Protter (2004), Çetin-Soner-Touzi (2010)).
- A few papers with some form of market impact (Lasry-Lions, Abergel-Loeper, Bouchard-Loeper)

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Recently, optimal execution met option pricing:

- L. C. Rogers, S. Singh, *The cost of illiquidity and its effects on hedging*. Mathematical Finance, 20(4), 597-615, 2010.
- O. Guéant, J. Pu, *Option pricing and hedging with execution costs and market impact*, Mathematical Finance, 2015.
- T. M. Li, R. Almgren, *Option hedging with smooth market impact*, MML, 2016.

Not a fantasy

Interesting quant note: *What does the saw-tooth pattern on US markets on 19 July 2012 tell us about the price formation process?*, C.-A. Lehalle et al., Crédit Agricole Cheuvreux Quant Note, Aug. 2012.

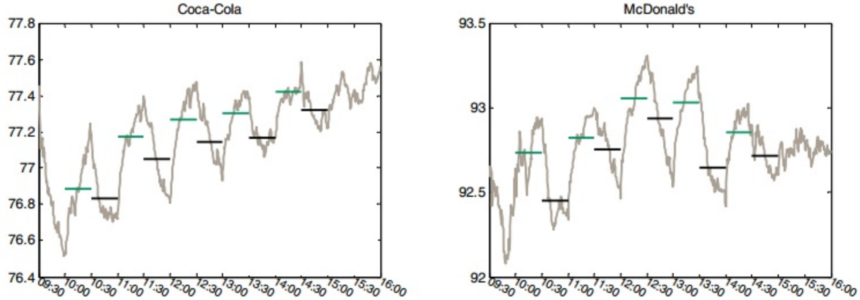
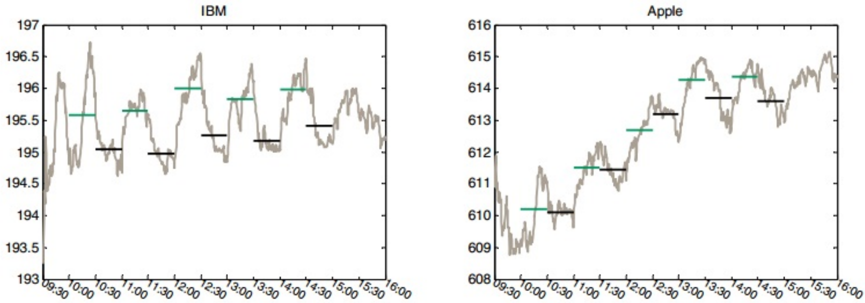


Figure: Saw tooth patterns on large caps

Not a fantasy

... Not small caps but major US stocks.



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Call option

Call/Put option on a stock with:

- Strike K
- Maturity T
- Nominal N (in shares)

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N matters because the introduction of execution costs and market impact makes the problem a non-linear one.

Notations

Model without permanent market impact for the sake of simplicity (permanent market impact corresponds to a change of variables in this model).

Framework in continuous time with 4 variables

- Time: t
- Number of shares: $q_t = q_0 + \int_0^t v_s ds$
- Price: $dS_t = \sigma dW_t$
- Cash: $dX_t = -v_t S_t dt - V_t L\left(\frac{v_t}{V_t}\right) dt$

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Remarks:

- q_0 is important here.
- V_t can be set to 0 at night!

Example: payoff of selling a call option (physical settlement)

Case 1 – the option is exercised:

- The trader has whatever is on his cash account X_T
- The trader receives KN
- The trader buys $(N - q_T)$ shares and deliver N shares

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- The trader buys $(N - q_T)$ shares and deliver N shares

The payoff in that case is:

$$\underbrace{X_T}_{\text{cash account}} + \underbrace{KN}_{\text{payment of the client}} - \underbrace{((N - q_T)S_T + \ell(N - q_T))}_{\text{cost of buying } N - q_T \text{ shares}}$$

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Payoff

$$X_T + q_T S_T + 1_{S_T \geq K} \left(N(K - S_T) - \ell(N - q_T) \right) - 1_{S_T < K} \ell(q_T)$$

Payoffs

Option	Position	Settlement	Terminal wealth
Call	Short	PS	$X_T + q_T S_T - N(S_T - K)_+ - \left(\ell(N - q_T)1_{S_T > K} + \ell(q_T)1_{S_T \leq K}\right)$
		CS	$X_T + q_T S_T - N(S_T - K)_+ - \ell(q_T)$
	Long	PS	$X_T + q_T S_T + N(S_T - K)_+ - \left(\ell(N + q_T)1_{S_T > K} + \ell(q_T)1_{S_T \leq K}\right)$
		CS	$X_T + q_T S_T + N(S_T - K)_+ - \ell(q_T)$
Put	Short	PS	$X_T + q_T S_T - N(S_T - K)_- - \left(\ell(N + q_T)1_{S_T < K} + \ell(q_T)1_{S_T \geq K}\right)$
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		CS	$X_T + q_T S_T + N(S_T - K)_- - \ell(q_T)$

Table: Terminal wealth for the different vanilla options.

Optimization Problem

Hereafter, we consider that the bank has sold a call option with physical settlement.

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Optimization Problem

The bank maximizes its expected utility:

$$\sup_{v \in \mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma Y_T\right)\right],$$

where $Y_T = X_T + q_T S_T$

$$+1_{S_T \geq K}\left(N(K - S_T) - \ell(N - q_T)\right) - 1_{S_T < K}\ell(q_T)$$

HJB Equation

The HJB equation associated with this stochastic optimal control problem is:

HJB equation

$$0 = -\partial_t u - \frac{1}{2}\sigma^2 \partial^2_{SS} u - \sup_{v \in \mathbb{R}} \left\{ v\partial_q u + \left(-vS - L\left(\frac{v}{V_t}\right) V_t \right) \partial_x u \right\}$$

with terminal condition:

$$\begin{aligned} u(T, x, q, S) \quad &= \quad -\exp\Bigg(-\gamma\Big(x + qS - 1_{S<K}\ell(q) \\ &\qquad + 1_{S\geq K}\left(N(K-S) - \ell(N-q)\right)\Big)\Bigg) \end{aligned}$$

Change of variables

We use the following change of variables:

Definition

We introduce θ by:

$$u(t,x,q,S) = -\exp\left(-\gamma\left(x + qS - \theta(t,q,S)\right)\right)$$

Change of variables

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Indifference price

$\theta(0,q_0,S_0)$ can be interpreted as the indifference price of the following contract:

- We write the call with the client
- We give $q_0 S_0$ to the client in cash
- The client gives us q_0 shares

PDE for θ

The PDE satisfied by θ is the following:

PDE

$$-\partial_t\theta - \frac{1}{2}\sigma^2\partial^2_{SS}\theta - \frac{1}{2}\gamma\sigma^2(\partial_S\theta - q)^2 + V_t H(\partial_q\theta) = 0$$

where H is as above $H(p) = \sup_{|\rho|\leq\rho_{\mathrm{m}}} \{p\rho - L(\rho)\}$.

Terminal condition

$\theta(T, q, S) = 1_{S\geq K}\left(N(S-K) + \ell(N-q)\right) + 1_{S<K}\ell(q)$

PDE

Interpretation of the PDE:

$$\underbrace{-\partial_t\theta - \frac{1}{2}\sigma^2\partial^2_{SS}\theta}_{\text{Bachelier PDE}} - \underbrace{\frac{1}{2}\gamma\sigma^2(\partial_S\theta - q)^2}_{\text{"Mishedge"}} + \underbrace{V_t H(\partial_q\theta)}_{\text{Execution costs}} = 0$$

Remark: This PDE is not an HJB equation. θ is rather the value function of a player in a zero-sum differential game.

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Remark: This PDE is not an HJB equation. θ is rather the value function of a player in a zero-sum differential game.

An optimal control is formally given by:

Optimal control

$$v^\star(t,q,S) = -V_t H'(\partial_q\theta(t,q,S)).$$

Reference scenario

- $S_0 = K = 45$ €.
- $\sigma = 0.6$ €$\cdot$day$^{-1/2}$ ($\approx 21\%$ annual volatility).
- $T = 63$ days.
- $V = 4\ 000\ 000$ shares$\cdot$day^{-1}.
- $N = 20\ 000\ 000$ shares.
- $L(\rho) = \eta|\rho|^{1+\phi}$ with $\eta = 0.1$ € $\cdot$shares$^{-1} \cdot$ day^{-1} and $\phi = 0.75$.

  That corresponds to 9 bps for a participation rate of 30% and 13 bps for a participation rate of 50%.
- $\gamma = 2 \cdot 10^{-7}$€$^{-1}$.
- ℓ corresponds to liquidation with POV at rate 50%.

Reference scenario

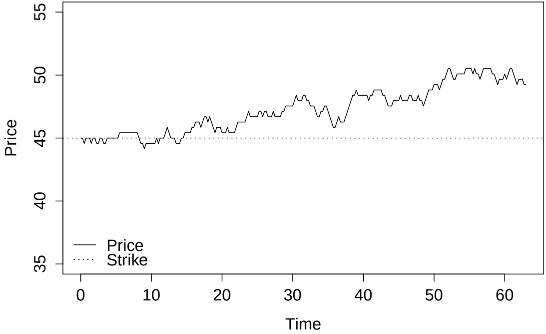

Figure: Reference scenario - Stock price

Reference scenario

2 numerical methods: a tree method and a finite difference scheme.
We see that we do not mean-revert around the usual Δ.

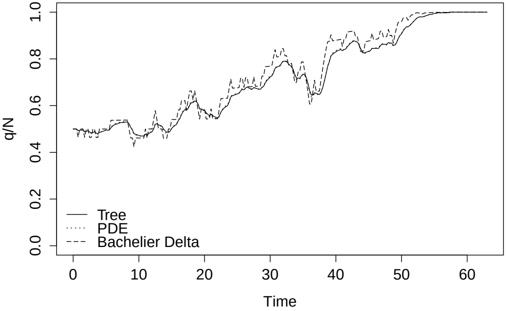

Figure: Reference scenario - Strategy

Reference scenario

Model/Method	Bachelier	Tree-Based approach	PDE approach
Price	1.900	2.060	2.067

Table: Prices of the call option for the two numerical methods.

We see the difference between the classical model and our model.

Importance of the initial position

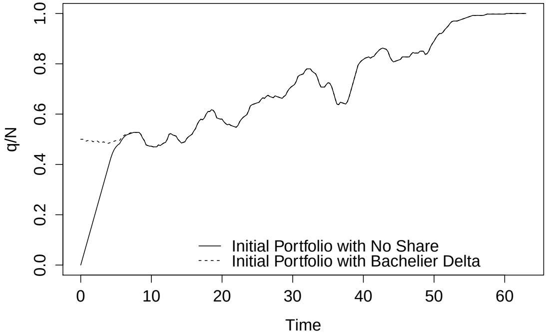

Figure: Optimal portfolio when $q_0 = 0$ and when a participation limit of 50% is imposed.

Execution Costs

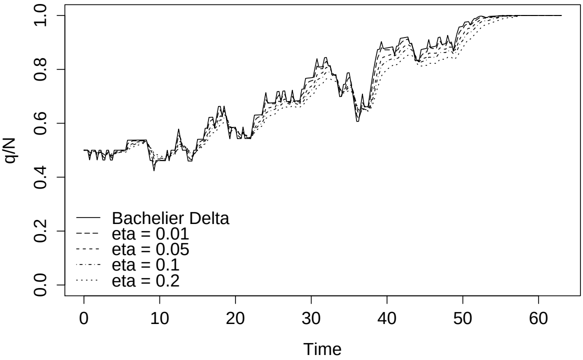

Figure: Optimal portfolio for different values of η.

Execution Costs

When η increases:

- The trajectories are smoother.
- They are closer to the position $0.5N$ to avoid round trips.

When $\eta \to 0$, we obtain the limiting case of Δ-Hedging.

The prices are given by:

η	0.2	0.1	0.05	0.01	0 (Bachelier)
Price of the call	2.14	2.06	2.01	1.94	1.90

- Prices are higher when η increases.

Price risk and risk aversion

- First risk (binary/digital): the trader will have to deliver either N shares or none. Being averse to this risk encourages the trader to stay close to a neutral portfolio with $q = 0.5N$.
- Second risk: price at which shares are bought/sold. Being averse to price risk encourages the trader to have a portfolio that evolves in the same direction as the price, as it is the case in the Bachelier model.

Price risk and risk aversion

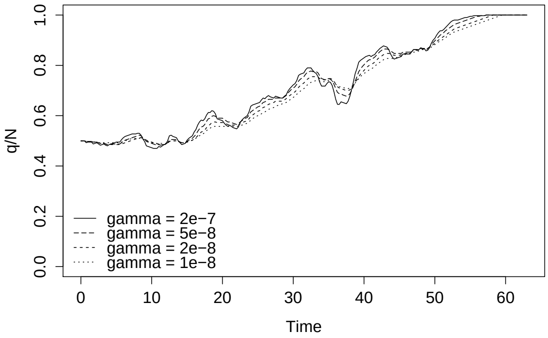

Figure: Optimal portfolio for different values of γ - 1

Price risk and risk aversion

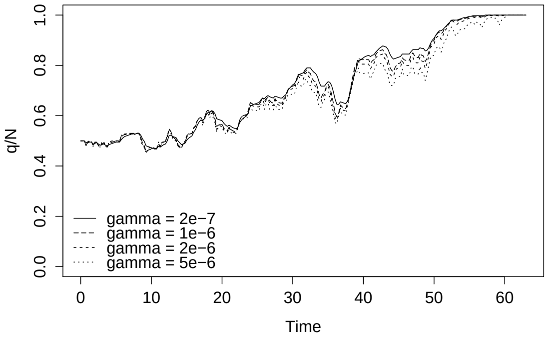

Figure: Optimal portfolio for different values of γ - 2

Price risk and risk aversion

The two effects are important. In terms of price there is a monotone dependence:

γ	$1 \cdot 10^{-8}$	$2 \cdot 10^{-8}$	$5 \cdot 10^{-8}$	$2 \cdot 10^{-7}$	$1 \cdot 10^{-6}$	$2 \cdot 10^{-6}$	$5 \cdot 10^{-6}$
Price of the call	1.955	1.968	1.994	2.060	2.207	2.308	2.521

Table: Prices of the call option for different values of γ .

Prices are increasing with γ. Prices also increase with σ.

Extensions

Many extensions are possible (see the paper)

- Interest rate r.
- Drift μ.
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Also in the paper

- Change of variables: $\tilde{\theta}(t, \tilde{q}) = \frac{1}{N}\theta(t, N\tilde{q})$.
- Comparison with Bachelier hedging with different frequencies.

ASR contracts

Introduction

Beyond option pricing

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- Let us now consider a problem with both execution issues and optional features:

Accelerated Share Repurchase contracts.

- ASR contracts are used by firms to buy back shares instead of paying dividends (*e.g.* tax reason).
- Instead of buying shares on the market, they ask a bank to do so and the contract includes an option for the bank (see below).

Introduction

Why ASR contracts?

Why not simply buying shares on markets?

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- Many repurchase programs are slowed down, postponed, or cancelled after announcement (because of unexpected shocks on prices for instance).

ASR contracts are mainly of two kinds: with fixed number of shares / with fixed notional.

ASR Contracts (fixed number of shares Q)

A bank is asked by a firm to repurchase a number $Q > 0$ of the firm's own shares.

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Shares are sometimes delivered to the firm over $[0, \tau]$ (not borrowed) or at time τ: in that case, the ASR is not anymore accelerated.

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ASR Contracts

Nature of the problem

- An optimal execution problem (shares are bought on the market by the bank) with usually huge nominal.
- An optimal stopping problem (Bermudan feature).
- An option pricing and hedging problem with Asian payoff.

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- An optimal execution problem (shares are bought on the market by the bank) with usually huge nominal.
- An optimal stopping problem (Bermudan feature).
- An option pricing and hedging problem with Asian payoff.

All these problems must be solved at the same time.

Remark: we ignore interest rates, repo and all financing issues in the model. This is why initial payments or initial delivery do not matter.

Setup of the model (fixed number of shares Q)

Discrete-time model

- $\delta t = 1$ day.
- $n = 0$ corresponds to $t = 0$.
- $T = N\delta t$ is the horizon of the ASR contract.

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- $T = N\delta t$ is the horizon of the ASR contract.

Dynamics I

- Q: number of shares to buy.
- $S_{n+1} = S_n + \sigma\sqrt{\delta t}\epsilon_{n+1}$: VWAP, with $(\epsilon_n)_{1\leq n\leq N}$ i.i.d.
- $A_n = \frac{1}{n}\sum_{k=1}^{n} S_k$: the average of daily VWAPs over the period $[0, n\delta t]$.
- $q_{n+1} = q_n + v_n\delta t$: the number of shares bought at time t_{n+1} $(q_0 = 0)$.

Setup of the model (continued)

Moreover, we consider a market with temporary market impact:

Dynamics II: cash spent

$$\begin{cases} X_0 & = & 0 \\ X_{n+1} & = & X_n + v_n S_{n+1}\delta t + L\left(\dfrac{v_n}{V_{n+1}}\right) V_{n+1}\delta t, \end{cases}$$

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where:

- $L : \mathbb{R} \to \mathbb{R}_+$ is strictly convex, increasing on $\mathbb{R}_+$, even, asymptotically super-linear.
- $(V_n)_n$ is the market volume process, assumed to be deterministic.

Setup of the model (continued)

Stopping time

- $\mathcal{N} \subset \{1, ..., N-1\}$ is the set of possible exercise times before expiry (usually, $\mathcal{N} = \{n_0, \ldots, N-1\}$).
- The exercise time $n^\star$ is a stopping time taking value in $\mathcal{N} \cup \{N\}$.

Setup of the model (continued)

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At and after the exercise time

- At time $t_{n^\star}$, $Q - q_{n^\star}$ shares remain to be bought.
- The pure optimal execution problem after time $n^\star$ is replaced by a proxy:

$$(Q - q_{n^\star})S_{n^\star} + \ell(Q - q_{n^\star}),$$

where ℓ is a penalty function (see BTP).

Objective function

We consider an expected utility framework:

Maximization problem

$$\sup_{(v,n^\star)\in\mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma\left(QA_{n^\star} - X_{n^\star} - (Q - q_{n^\star})S_{n^\star} - \ell(Q - q_{n^\star})\right)\right)\right],$$

where γ is the absolute risk aversion of the bank.

Bellman characterization setup

The associated dynamic value function

$$u_n(x, q, S, A) = \sup_{(v, n^\star)}$$

$$\mathbb{E}\left[-\exp\left(-\gamma\left(QA_{n^\star}^{n,A,S} - X_{n^\star}^{n,x,v} - (Q - q_{n^\star}^{n,q,v})S_{n^\star}^{n,S} - \ell(Q - q_{n^\star}^{n,q,v})\right)\right)\right]$$

Finally, we define:

$$\tilde{u}_{n,n+1}(x, q, S, A) = \sup_{v \in \mathbb{R}} \mathbb{E}\left[u_{n+1}\left(X_{n+1}^{n,x,v}, q_{n+1}^{n,q,v}, S_{n+1}^{n,S}, A_{n+1}^{n,A,S}\right)\right].$$

Bellman characterization

Dynamic programming principle

- $u_N(X,q,S,A) =$
  $-\exp\left(-\gamma\left(QA - X - (Q-q)S - \ell(Q-q)\right)\right)$
- for $n \in \mathcal{N}$,

$$u_n(X,q,S,A) = \max\left\{\tilde{u}_{n,n+1}\left(x,q,S,A\right),\right.$$

$$\left.-\exp\left(-\gamma\left(QA - X - (Q-q)S - \ell(Q-q)\right)\right)\right\}$$

- for $n \notin \mathcal{N}$ and $n \neq N$:

$$u_n(X,q,S,A) = \tilde{u}_{n,n+1}\left(X,q,S,A\right)$$

Main result

Proposition (Change of variables)

For $n \geq 1$, $u_n(x,q,S,A)$ can be written as

$$u_n(x,q,S,A) = -\exp\left(-\gamma\left(Y - \theta_n\left(q, \frac{S-A}{\sigma\sqrt{\delta t}}\right)\right)\right),$$

where $Y = Q(A-S) - X + qS$ and $\theta_n(q,Z)$ is equal to:

$$\inf_{(v,n^\star)} \frac{1}{\gamma} \log\left(\mathbb{E}\left[\exp\left(\gamma\left(\sigma\sqrt{\delta t}\left(\sum_{j=n}^{n^\star-1}\left(\frac{j}{n^\star}Q - q_j\right)\epsilon_{j+1}\right.\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.\left.-\left(1-\frac{n}{n^\star}\right)QZ\right) + \sum_{j=n}^{n^\star-1} L\left(\frac{v_j}{V_{j+1}}\right)V_{j+1}\delta t + \ell(Q - q_{n^\star})\right)\right)\right]\right).$$

Bellman equation for θ_n

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- for $n = N$: $\theta_n(q, Z) = \ell(Q - q)$,
- for $n \in \mathcal{N}$: $\theta_n(q, Z) = \min\left\{\tilde{\theta}_{n,n+1}(q, Z), \ell(Q - q)\right\}$,
- for $n \notin \mathcal{N}$: $\theta_n(q, Z) = \tilde{\theta}_{n,n+1}(q, Z)$,

where $\tilde{\theta}_{n,n+1}$ is equal to:

$$\inf_{v \in \mathbb{R}} \frac{1}{\gamma} \log\left(\mathbb{E}\left[\exp\left(\gamma\left(\sigma\sqrt{\delta t}\left(\left(\frac{n}{n+1}Q - q\right)\epsilon_{n+1} - \frac{Q}{n+1}Z\right)\right.\right.\right.\right.$$

$$\left.\left.\left.\left. + L\left(\frac{v}{V_{n+1}}\right)V_{n+1}\delta t + \theta_{n+1}\left(q + v\delta t, \frac{n}{n+1}\left(Z + \epsilon_{n+1}\right)\right)\right)\right)\right]\right).$$

Analysis of θ_n

Our change of variables can be interpreted easily. We recall that $\theta_n(q, Z)$ is equal to:

$$\inf_{(v,n^\star)} \frac{1}{\gamma} \log\left(\mathbb{E}\left[\exp\left(\gamma\left(\sigma\sqrt{\delta t}\underbrace{\left(\sum_{j=n}^{n^\star-1}\left(\frac{j}{n^\star}Q - q_j\right)\epsilon_{j+1}\right.}_{\text{risk term}} - \underbrace{\left.\left(1-\frac{n}{n^\star}\right)QZ\right)}_{Z\text{ term}}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\underbrace{\sum_{j=n}^{n^\star-1} L\left(\frac{v_j}{V_{j+1}}\right)V_{j+1}\delta t}_{\text{liquidity term before exercise}} + \underbrace{\ell(Q - q_{n^\star})}_{\text{liquidity and risk term after exercise}}\right)\right)\right]\right).$$

Analysis of θ_n

The previous formula helps to understand the effects at stake:

The risk term

- The risk term measures the risk associated to a deviation from a straight-line strategy.
- If the bank buys Q shares evenly until a given exercise date (or until T), then the risk is indeed perfectly hedged.
- But to benefit from the option contract, the bank will not follow this strategy.

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The Z-term

- If the price goes down, then there is an incentive to exercise to benefit from the difference between A and S...
- ... but this incentive depends on q (see below).

Analysis of θ_n

The ℓ term

- Before time $n^\star$, the execution process is partially hedged (this is the risk term)
- After time $n^\star$, the execution process is not hedged (the risk is in the ℓ-term).
- Hence, there is an incentive to delay exercise if we have still a large number of shares to buy.

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- After time $n^\star$, the execution process is not hedged (the risk is in the ℓ-term).
- Hence, there is an incentive to delay exercise if we have still a large number of shares to buy.

The consequence is that when S goes down the bank should accelerate the execution (buying) process, but not too much (because of execution costs).

Indifference price of ASR

One can easily prove that u_0 does not depend on A and that:

$$u_0(X=0, q=0, S_0) = -\exp\left(\gamma \inf_{v\in\mathbb{R}} \left\{ L\left(\frac{v}{V_1}\right) + \theta_1(v\delta t, 0)\right\}\right).$$

Hence, the amount of cash that makes the bank indifferent between signing and not signing the ASR contract is:

$$\Pi = \inf_{v\in\mathbb{R}} \left\{ L\left(\frac{v}{V_1}\right) + \theta_1(v\delta t, 0)\right\}.$$

This is the indifference price.

Indifference price of ASR

The sign of the price Π is important:

- If Π is negative, it means that the gain associated to the option is larger than the execution costs.
- If Π is positive, it means that the option does not compensate execution costs.

In practice, deals occur only in the first case, and competition between banks is through a discount/rebate on the average price A.
Remark: equations are different with a discount.

Discussion

Optimal strategy – optimal exercise time

- The optimal strategy only depends on q and Z
- Exercise if $Z_n \leq Z_n^{\text{exec}}(q)$.

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Extensions

- We can add permanent market impact.
- We can add participation constraints.
- Continuous time trading strategy (see also another paper by Jaimungal *et al.*).

Numerical scheme

Tree method

We consider a pentanomial tree model for innovations $(\epsilon_n)_{n\geq 1}$:

$$\epsilon_n = \begin{cases} +2 & \text{with probability } \frac{1}{12} \\ +1 & \text{with probability } \frac{1}{6} \\ 0 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{6} \\ -2 & \text{with probability } \frac{1}{12} \end{cases}$$

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These values for the distribution of ϵ_n are chosen to match the first four moments of the standard normal distribution, i.e. we have:

$$\mathbb{E}\left[\epsilon_n\right] = 0,\ \mathbb{E}\left[\epsilon_n^2\right] = 1,\ \mathbb{E}\left[\epsilon_n^3\right] = 0,\ \mathbb{E}\left[\epsilon_n^4\right] = 3.$$

Numerical scheme

- Each node of the tree corresponds to a couple (n, Z) and we associate an array for q to each node.
- The tree is not recombinant in the classical sense.
- However $nZ_n + n(n-1)$ is an integer between 0 and $2n(n-1)$.
- Hence the tree has a number of nodes that is a cubic function of N.

Reference case

- $S_0 = 45$ €
- $\sigma = 0.6$ €$\cdot$day$^{-1/2}$, which corresponds to an annual volatility approximately equal to 21%.
- $T = 63$ trading days
- $V = 4\ 000\ 000$ stocks$\cdot$ day^{-1}
- $Q = 20\ 000\ 000$ stocks
- $L(\rho) = \eta|\rho|^{1+\phi}$ with $\eta = 0.1$ €$\cdot$stock$^{-1} \cdot$ day^{-1} and $\phi = 0.75$
- $\gamma = 2.5 \cdot 10^{-7}$ €$^{-1}$.
- $\ell(q)$ corresponds to execution at participation rate 25% after the exercise date.

The set of possible exercise dates is $\mathcal{N} = [22, 62] \cap \mathbb{N}$.

Price trajectory and optimal strategy I

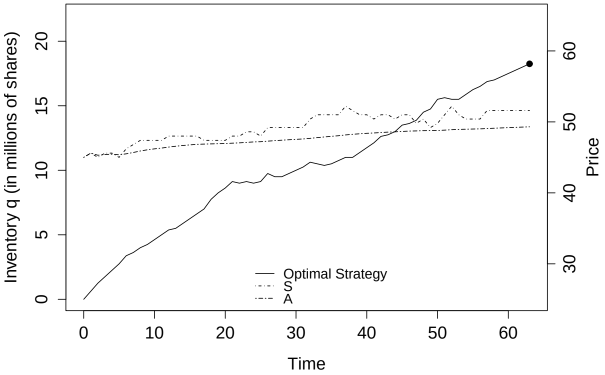

Figure: Optimal Strategy when price goes up.

Price trajectory and optimal strategy I

In that case:

- Exercise at terminal time.
- Minimizing execution costs by trading almost in straight line.
- When S decreases, acceleration of the buying process.
- When S increases, the buying process slows down or even turns into a selling process (for hedging purposes).

Price trajectory and optimal strategy II

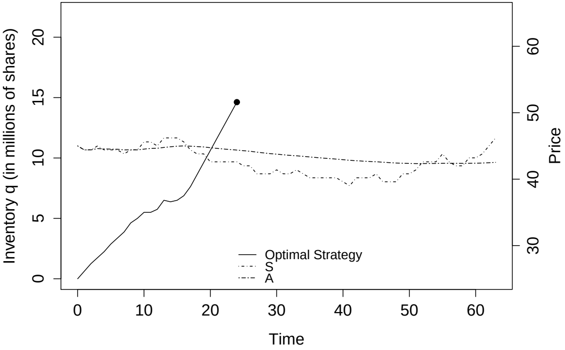

Figure: Optimal Strategy when price goes down.

Price trajectory and optimal strategy II

In that case:

- Exercise almost as soon as possible (to benefit from $A - S$).
- As S is below A, acceleration of the buying process to buy a lot before exercising.

Price trajectory and optimal strategy III

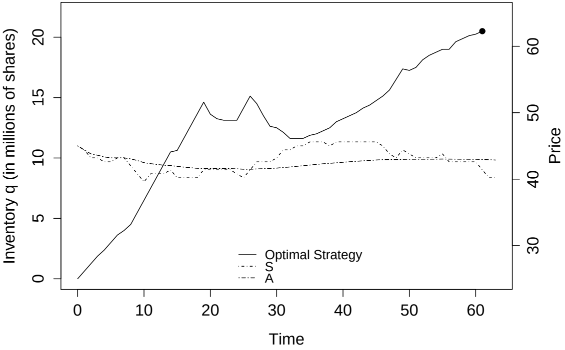

Figure: Optimal Strategy when price oscillates.

Price trajectory and optimal strategy III

- The effects at stake are the same as above.
- The indifference price obtained is: $-10031490 = -1.11\% QS_0 < 0$
- If we constrain the strategies to be buy-only strategies, we get: $-1.08\% QS_0 < 0$

Price trajectory and optimal strategy III

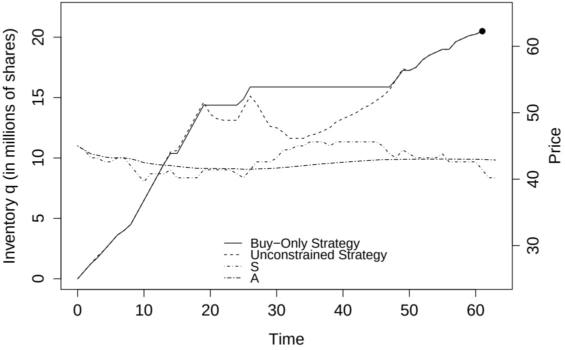

Figure: Optimal Buy-only Strategy when price oscillates.

Effect of execution costs, case III

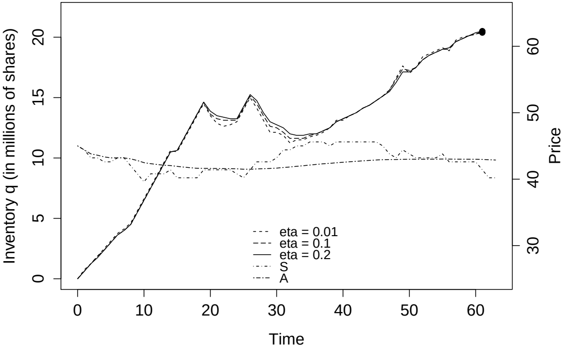

Figure: Optimal strategies for different values of η for price trajectory III

Effect of execution costs

Utility indifference price of ASR contracts for different values of η:

η	0.01	0.1	0.2
$\frac{\Pi}{QS_0}$	-1.18%	-1.11%	-1.05%

The less liquid the stock, the less round trips on the stock and the less the bank can give back as a discount to the firm.

Effect of risk aversion, case I

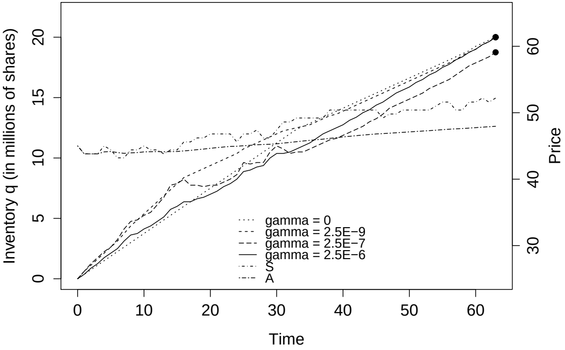

Figure: Optimal strategies for different values of γ for price trajectory I

Effect of risk aversion

For risk aversion there are several effect at stake, and the shape of strategies is not monotonic in γ. For instance, a high γ leads at the same time to a curve closer to a straight line to hedge, and to sharp increases in q to exercise with less to execute without hedge.

Effect of risk aversion

For risk aversion there are several effect at stake, and the shape of strategies is not monotonic in γ. For instance, a high γ leads at the same time to a curve closer to a straight line to hedge, and to sharp increases in q to exercise with less to execute without hedge. However, the influence of γ on the price is clear.
Utility indifference price of ASR contracts for different values of γ:

γ	0	$2.5 \cdot 10^{-9}$	$2.5 \cdot 10^{-7}$	$2.5 \cdot 10^{-6}$
$\frac{\Pi}{QS_0}$	-1.39%	-1.38%	-1.18%	-0.44%

The more risk averse, the less discount it will propose to the firm.

The fixed notional case – Objective function

The maximization problem in the fixed notional case becomes:

Objective function

$$\mathbb{E}\left[-\exp\left(-\gamma\left(F - X_{n^\star} - \left(\frac{F}{A_{n^\star}} - q_{n^\star}\right)S_{n^\star} - \ell\left(\frac{F}{A_{n^\star}} - q_{n^\star}\right)\right)\right)\right]$$

- Going from 5 to 3 variables is now impossible, as (S, A) cannot be reduced to $S - A$
- However, X can still be factored out.

The fixed notional case – Comments

- The above numerical method cannot be applied.
- We used a method with a tree for S, a grid for (q, A) at each node... and interpolation with splines (for A) whenever necessary.
- Perfect hedging with straight-line strategies do not exist anymore.
- On all numerical examples: more profitable for the bank to write fixed notional contract. Not as simple as convexity, though...

Example - Case I

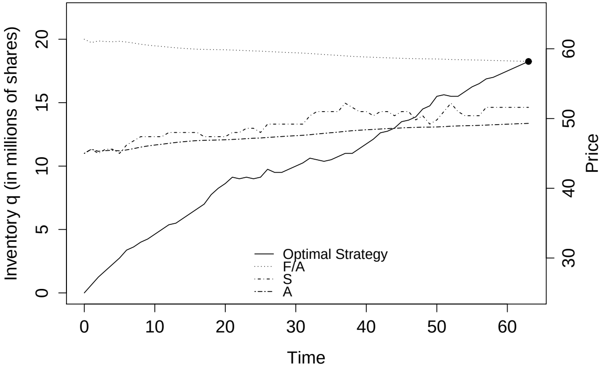

Figure: Optimal Strategy when price goes up (Fixed notional).

Final remarks

Conclusion

- Optimal execution tools can be used beyond optimal scheduling:
  - Block trade pricing.
  - Option hedging.
  - The management of complex execution contracts with optional features.

Conclusion

- Optimal execution tools can be used beyond optimal scheduling:
  - Block trade pricing.
  - Option hedging.
  - The management of complex execution contracts with optional features.
- Theoretical work remains to be done $\rightarrow$ *e.g.*, non-linear PDEs.

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Asset management, portfolio choice, portfolio transition.

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Asset management, portfolio choice, portfolio transition.

Teasing for Lecture 3.

End of Lecture 2

Thank you. Questions?

Lecture 3: Asset management with execution costs.

Introduction

Liquidity issues are everywhere

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Lecture 1

I have introduced the Almgren-Chriss model:

- Initial Almgren-Chriss (quadratic) model in discrete time.
- Generalized Almgren-Chriss model in continuous time.
- Use of the Almgren-Chriss in the brokerage industry (IS, TC, and POV orders).

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Lecture 2

The use of the Almgren-Chriss model for pricing and hedging:

- Block trade pricing.
- Pricing and hedging of vanilla options (physical/cash settlement).
- Pricing and hedging of Accelerated Share Repurchase (ASR) contracts.

Liquidity issues are everywhere

Other domains of finance are concerned with liquidity issues:

- Risk management.
- Market making.
- Asset management – return / risk (volatility, skew, kurtosis) $+$ liquidity.

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- Portfolio choice and asset management with execution costs.

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Lecture 3

- Portfolio choice and asset management with execution costs.
- Bayesian learning (on the drift) $+$ stochastic optimal control.

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Mixing learning and optimal control is a (trendy) idea that goes beyond financial applications.

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- Bayesian learning (on the drift) + stochastic optimal control.

Mixing learning and optimal control is a (trendy) idea that goes beyond financial applications.

Most of the original content of today's lecture is in the paper "Portfolio choice under drift uncertainty: a Bayesian learning and stochastic optimal control approach" by OG and J. Pu.

Asset management and portfolio choice: reminders

A bit of history

- Markowitz and its efficient frontier.
- Tobin and the separation theorem.
- Sharpe and others with the CAPM.
- Merton's problem (with and without consumption).
  $\rightarrow$ Dynamic portfolio choice.
- APT + Fama-French.
- Black-Litterman (Markowitz + CAPM).
- ...

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We will focus on Merton's problem without consumption.

Classical problem with 2 assets

2 assets

- Risk-free asset. Interest rate r.
- Risky asset:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad \sigma > 0.$$

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$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad \sigma > 0.$$

Portfolio dynamics

$$\begin{aligned} dV_t &= ((\mu - r)\,\theta_t V_t + rV_t)\,dt + \sigma\theta_t V_t dW_t \\ &= ((\mu - r)\,M_t + rV_t)\,dt + \sigma M_t dW_t \end{aligned}$$

- θ: proportion of the portfolio invested in the risky asset.
- M: amount invested in the risky asset.

Classical problem with 2 assets

Objective function

$$\sup_{\theta \in \mathcal{A}} \mathbb{E}\left[U\left(V_T^{0,V_0,\theta}\right)\right],$$

where $\mathcal{A}$ is the set of admissible strategies (see paper).

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Objective function

$$\sup_{\theta \in \mathcal{A}} \mathbb{E}\left[U\left(V_T^{0,V_0,\theta}\right)\right],$$

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Two important cases

- CARA: $U(V) = -\exp(-\gamma V)$
- CRRA:

$$U(V) = \begin{cases} \dfrac{V^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1, \\ \log(V) & \text{if } \gamma = 1. \end{cases}$$

The PDE approach

The PDE approach

Value function

$$v(t, V) = \sup_{\theta \in \mathcal{A}_t} \mathbb{E}\left[U\left(V_T^{t,V,\theta} \right) \right].$$

The PDE approach

Value function

$$v(t,V) \quad = \quad \sup_{\theta \in \mathcal{A}_t} \mathbb{E}\left[U\left(V_T^{t,V,\theta}\right)\right].$$

HJB equation

$$\begin{aligned} -\partial_t u(t,V) - \sup_{\theta}\Big\{ & ((\mu - r)\theta + r)\, V \partial_V u(t,V) \\ & + \frac{1}{2}\sigma^2\theta^2 V^2 \partial^2_{VV} u(t,V)\Big\} \quad = \quad 0, \end{aligned}$$

with terminal condition

$$u(T,V) \quad = \quad U(V).$$

CRRA case

Ansatz

$$u(t, V) = \frac{\left(e^{r(T-t)} V\right)^{1-\gamma}}{1-\gamma} \exp(g(t)).$$

CRRA case

Ansatz

$$u(t,V) = \frac{\left(e^{r(T-t)}V\right)^{1-\gamma}}{1-\gamma}\exp(g(t)).$$

Equation for g

The HJB equation becomes:

$$g'(t) + (1-\gamma)\sup_{\theta}\left((\mu - r)\theta - \frac{1}{2}\gamma\sigma^2\theta^2\right) = 0, \quad g(T) = 0.$$

CRRA case

Solution of the HJB equation

$$u(t,V) = \frac{\left(e^{r(T-t)}V\right)^{1-\gamma}}{1-\gamma} \exp\left[\frac{1-\gamma}{2\gamma\sigma^2}(\mu - r)^2(T-t)\right].$$

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Optimizer

$$\theta^{\star} = \frac{\mu - r}{\gamma\sigma^2}$$

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The verification approach leads to $u = v$ and $\theta^\star$ is optimal among L^2 adapted processes with linear growth in W.

CARA case

Ansatz

$$u\left(t, V\right) = -\exp\left[-\gamma\left(e^{r(T-t)}V + g(t)\right)\right].$$

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Equation for g

The HJB equation becomes:

$$g'(t) + \sup_M \left((\mu - r)Me^{r(T-t)} - \frac{1}{2}\gamma\sigma^2 M^2 e^{r(T-t)}\right) = 0, g(T) = 0$$

CARA case

Solution of the HJB equation

$$u(t, V) = -\exp\left[-\gamma\left(e^{r(T-t)}V + \frac{1}{2\gamma\sigma^2}(T-t)(\mu-r)^2\right)\right].$$

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Optimizer

$$M^\star = \theta^\star V \quad = \quad e^{-r(T-t)}\frac{\mu-r}{\gamma\sigma^2}$$

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The dual/martingale approach

Martingale approach – Principle I

Introduction of a martingale measure $\mathbb{Q}$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T = e^{-\frac{\mu-r}{\sigma}W_T - \frac{1}{2\sigma^2}(\mu-r)^2 T},$$

$$W_t^{\mathbb{Q}} = W_t + \frac{\mu - r}{\sigma},$$

such that

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}. \\ dV_t &= rV_t + \sigma\theta V_t dW_t^{\mathbb{Q}}. \end{aligned}$$

Martingale approach – Principle II

Concavity of U

$$\begin{aligned}\mathbb{E}\left[U\left(V_T^{0,V_0,\theta}\right)\right] \quad &\leq \quad \mathbb{E}\left[U\left(V_T^{0,V_0,\theta^\star}\right)\right] \\ &+\mathbb{E}\left[U'\left(V_T^{0,V_0,\theta^\star}\right)\left(V_T^{0,V_0,\theta} - V_T^{0,V_0,\theta^\star}\right)\right]\end{aligned}$$

Martingale approach – Principle II

Concavity of U

$$
\begin{aligned}
\mathbb{E}\left[U\left(V_T^{0,V_0,\theta}\right)\right] &\leq \mathbb{E}\left[U\left(V_T^{0,V_0,\theta^\star}\right)\right] \\
&\quad +\mathbb{E}\left[U'\left(V_T^{0,V_0,\theta^\star}\right)\left(V_T^{0,V_0,\theta} - V_T^{0,V_0,\theta^\star}\right)\right] \\
&\leq \mathbb{E}\left[U\left(V_T^{0,V_0,\theta^\star}\right)\right] \\
&\quad +\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{Z_T}U'\left(V_T^{0,V_0,\theta^\star}\right)\left(V_T^{0,V_0,\theta} - V_T^{0,V_0,\theta^\star}\right)\right].
\end{aligned}
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Martingale approach – Principle II

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\end{aligned}
$$

If $U'(V_T^{0,V_0,\theta^\star}) = cZ_T e^{-rT}$, then $\theta^\star$ is optimal!

Martingale approach – Identification I

Choice of c

We want

$$V_T^{0,V_0,\theta^\star} = U'^{-1}\left(cZ_T e^{-rT}\right)$$

and so

$$V_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[U'^{-1}\left(cZ_T e^{-rT}\right)\right].$$

This defines c (when a solution exists).

Martingale approach – Identification II

Finding $\theta^\star$

By definition

$$V_t^{0,V_0,\theta^\star} = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[U'^{-1}\left(cZ_T e^{-rT}\right)|\mathcal{F}_t\right].$$

and

$$dV_t^{0,V_0,\theta^\star} = rV_t^{0,V_0,\theta^\star}dt + \sigma\theta^\star V_t^{0,V_0,\theta^\star}dW_t^{\mathbb{Q}}$$

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$\theta^\star$ can be identified:

Martingale approach – Identification II

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$\theta^\star$ can be identified:

- **(theoretically) by the martingale representation theorem,**
- **(practically) by computing the above expected value (if U'^{-1} permits it) and applying Ito's formula.**

Remarks

Advantages and drawbacks

- The martingale method can be used for a large class of utility functions U.
- The martingale method requires to have... martingales (not the case with transaction costs for instance).

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- The martingale method can be used for a large class of utility functions U.
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Last remark: in both cases, we can easily generalize to $d > 1$ risky assets.

Appendix: Gaussian prices

Gaussian prices instead of Gaussian returns

2 assets

- Risk-free asset. No interest (to simplify).
- Risky asset:

$$dS_t = \mu dt + \sigma dW_t, \qquad \sigma > 0.$$

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Portfolio dynamics

$$dV_t = \mu N_t dt + \sigma N_t dW_t,$$

where N_t is the number of shares in the portfolio at date t.

Gaussian prices instead of Gaussian returns

Objective function

$$\sup_{N\in\mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma V_T^{0,V_0,N}\right)\right],$$

where $\mathcal{A}$ is the set of admissible strategies.

Gaussian prices instead of Gaussian returns

Objective function

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$$v(t, V) = \sup_{N \in \mathcal{A}_t} \mathbb{E}\left[-\exp\left(-\gamma V_T^{t,V,N}\right)\right]$$

Gaussian prices instead of Gaussian returns

HJB equation

$$-\partial_t u(t,V) - \sup_N \left\{ \mu N \partial_V u(t,V) + \frac{1}{2}\sigma^2 N^2 \partial^2_{VV} u(t,V) \right\} \;=\; 0,$$

with terminal condition

$$u(T,V) \;=\; -\exp(-\gamma V).$$

Change of variables

Ansatz

$$u\,(t, V) \quad = \quad -\exp\left[-\gamma\,(V + g(t))\right].$$

Change of variables

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$$u(t, V) \quad = \quad -\exp\left[-\gamma\left(V + g(t)\right)\right].$$

Equation for g

The HJB equation becomes:

$$g'(t) + \sup_N \left(\mu N - \frac{1}{2}\gamma\sigma^2 N^2\right) = 0, g(T) = 0$$

Solution

Solution of the HJB equation

$$u(t, V) = -\exp\left[-\gamma\left(V + \frac{1}{2\gamma\sigma^2}(T-t)\mu^2\right)\right].$$

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Optimizer

$$N^\star = \frac{\mu}{\gamma\sigma^2}.$$

Mixing Almgren-Chriss and Merton's problem

Mixing Almgren-Chriss and Merton

Almgren-Chriss framework

- Time: t.
- Number of shares: $q_t = q_0 + \int_0^t v_s ds$.
- Price: $dS_t = \mu dt + \sigma dW_t$.
- Cash: $dX_t = -v_t S_t dt - V_t L\left(\frac{v_t}{V_t}\right) dt, \qquad X_0 = 0$.

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Optimization problem

$$\sup_{(v_t)_t \in \mathcal{A}} \mathbb{E}\left[-\exp(-\gamma(X_T + q_T S_T - \ell(q_T)))\right], \qquad T \ \text{ fixed}$$

$$\mathcal{A} = \left\{ (v_t)_{t\in[0,T]} \text{ prog mes }, \int_0^T |v_t| dt \in L^\infty \right\}$$

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Remark: L satisfies the same assumptions as in Lecture 1 and ℓ is convex.

HJB and HJ equations

HJB Equation

The HJB equation associated with this stochastic optimal control problem is:

HJB equation

$$0 = \partial_t u + \mu \partial_S u + \frac{1}{2}\sigma^2 \partial^2_{SS} u + \sup_{v \in \mathbb{R}} \left\{ v \partial_q u + \left(-vS - L\left(\frac{v}{V_t}\right) V_t \right) \partial_x u \right\}$$

with terminal condition:

$$u(T, x, q, S) \quad = \quad -\exp(-\gamma(x + qS - \ell(q)))$$

Change of variables

Ansatz

$$u(t, x, q, S) = -\exp\left(-\gamma\left(x + qS - \theta(t, q)\right)\right)$$

Change of variables

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$$u(t, x, q, S) = -\exp\left(-\gamma\left(x + qS - \theta(t, q)\right)\right)$$

The PDE satisfied by θ is the following:

PDE

$$\partial_t \theta - \mu q + \frac{1}{2}\gamma\sigma^2 q^2 - V_t H(\partial_q \theta) = 0$$

with $\theta(T, q) = \ell(q)$.

Change of variables

Ansatz

$$u(t,x,q,S) \quad = \quad -\exp\left(-\gamma\left(x + qS - \theta(t,q)\right)\right)$$

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with $\theta(T,q) = \ell(q)$.

Optimal control

$$v^\star(t,q) = V_t H'(-\partial_q\theta(t,q))$$

Variational problem

Towards a variational problem

Expression of X_T

$$X_T + q_T S_T - \ell(q_T)$$

$$= X_0 + q_0 S_0 + \mu \int_0^T q_t + \sigma \int_0^T q_t dW_t - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt - \ell(q_T).$$

Towards a variational problem

Expression of X_T

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$$= X_0 + q_0 S_0 + \mu \int_0^T q_t + \sigma \int_0^T q_t dW_t - \int_0^T V_t L\left(\frac{v_t}{V_t}\right) dt - \ell(q_T).$$

By taking the Laplace transform (for v deterministic – using the same trick as for the AC model), the problem boils down to the following minimization problem:

Minimization problem

$$\inf_{q \in W^{1,1}(0,T), q(0)=q_0} \mathcal{I}(q),$$

where

$$\mathcal{I}(q) = \int_0^T \left(V_s L\left(\frac{\dot{q}(s)}{V_s}\right) - \mu q(s) + \frac{1}{2}\gamma\sigma^2 q^2(s) \right) ds + \ell(q(T)).$$

Variational approach

Theorem (Existence and uniqueness of a minimizer)

There exists a unique minimizer $q \in \{q \in W^{1,1}(0,T), q(0) = q_0\}$ *of* $\mathcal{I}$.

Variational approach

Theorem (Existence and uniqueness of a minimizer)

There exists a unique minimizer $q \in \{q \in W^{1,1}(0,T), q(0) = q_0\}$ of $\mathcal{I}$.

The problem can be solved using Euler-Lagrange equations or Hamiltonian equations.

Hamiltonian characterization

$$\begin{cases} \dot{p}(t) & = & -\mu + \gamma\sigma^2 q(t) \\ \dot{q}(t) & = & V_t H'(p(t)) \end{cases} \qquad q(0) = q_0, \quad p(T) = -\ell'(q(T)).$$

Remarks

- The system can only be solved numerically in general.
- It is interesting to see that the steady state corresponds to

$$q = \frac{\mu}{\gamma \sigma^2}.$$

- The system can be solved in closed form in the original (quadratic) Almgren-Chriss setting:

$$L(\rho) = \eta \rho^2, \qquad H(p) = \frac{p^2}{4\eta},$$

$$\ell(q) = \frac{1}{2} K q^2, \qquad V_t = V.$$

Equation in the quadratic case

Elliptic equation

The problem boils down to an elliptic equation:

$$q''(t) - \underbrace{\frac{\gamma\sigma^2 V}{2\eta}}_{=\alpha^2} q(t) = -\frac{\mu V}{2\eta},$$

with boundary conditions

$$q(0) = q_0, \qquad q'(T) = -\frac{KV}{2\eta} q(T).$$

Solution in the quadratic case

Solution

$$q(t) = \frac{\mu}{\gamma\sigma^2} + \left(q_0 - \frac{\mu}{\gamma\sigma^2}\right)\cosh(\alpha t) + B\sinh(\alpha t),$$

where

$$B = -\frac{\alpha\left(q_0 - \frac{\mu}{\gamma\sigma^2}\right)\sinh(\alpha t) + \frac{KV}{2\eta}\frac{\mu}{\gamma\sigma^2} + \frac{KV}{2\eta}\left(q_0 - \frac{\mu}{\gamma\sigma^2}\right)\cosh(\alpha T)}{\alpha\cosh(\alpha T) + \frac{KV}{2\eta}\sinh(\alpha T)}.$$

Examples

- $\mu = 0.01$ €$\cdot$day^{-1}.
- $\sigma = 0.6$ €$\cdot$day$^{-1/2}$.
- $T = 10$ days.
- $V = 4\ 000\ 000$ shares$\cdot$day^{-1}.
- $L(\rho) = \eta|\rho|^2$ with $\eta = 0.15$ € $\cdot$shares^{-1} $\cdot$ day^{-1}.
- $\gamma = 2 \cdot 10^{-7}$€$^{-1}$.

Examples

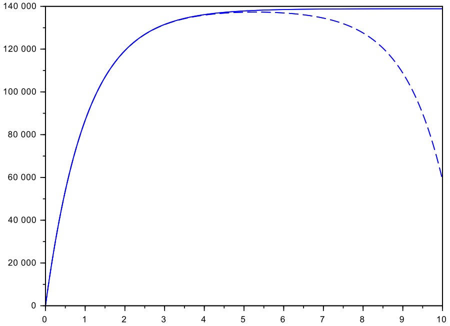

Figure: Optimal strategies for $\ell(q) = 0$ and $\ell(q) = 5 \cdot 10^{-8} q^2$.

Remarks

- Final penalty may not be the right way to penalize illiquidity.
- A running penalty has the same effect as increasing risk aversion or volatility.

Remarks

- Final penalty may not be the right way to penalize illiquidity.
- A running penalty has the same effect as increasing risk aversion or volatility.
- Possibility to consider portfolio transition:

$$\left\{\begin{array}{rcl} \dot{p}(t) & = & -\mu + \gamma\sigma^2 q(t) \\ \dot{q}(t) & = & V_t H'(p(t)) \end{array}\right. \qquad q(0) = q_0,$$

and

$$q(T) = q_{\text{target}} \quad \text{(portfolio transition problem)}$$

or

$$p(T) = -K(q(T) - q_{\text{target}}) \quad \text{(relaxed portfolio transition problem).}$$

Generalization

The problem can be generalized to a multi-asset portfolio (as the initial Almgren-Chriss model). In that case:

Hamiltonian characterization

$$\begin{cases} \dot{p}(t) & = & -\mu + \gamma \Sigma q(t) \\ \dot{q}^i(t) & = & V_t^i H^{i\prime}(p^i(t)), \forall i \end{cases} \qquad q(0) = q_0, p(T) = -\nabla \ell(q(T)),$$

Learning meets optimal control

Introduction

Stochastic optimal control

Stochastic optimal control is often used in finance for solving dynamic optimization problems.

Tools

- Dynamic programming principle.
- Hamilton-Jacobi-Bellman equation (PDE).
- Dual martingale methods.

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- Dual martingale methods.

Most common applications

- Portfolio choice / Asset management.
- Super-replication.
- Optimal execution.
- Market making strategies.

Bayesian learning

Bayesian learning

- Unknown parameter(s) $\rightarrow$ prior belief / prior distribution.
- Bayes' rule to update belief as information becomes available.
- Conjugate priors help a lot.

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- Conjugate priors help a lot.

Bayesian learning is a **forward** process whereas stochastic optimal control is based on a **backward** reasoning.

$\rightarrow$ **What happens when we learn and anticipate we will go on learning?**

Is it a new idea?

People have always learnt and controlled at the same time...
but they seldom anticipated the fact that they learn: they are often time-inconsistent!

Explore vs. exploit

- Very common in many fields where there is an explore/exploit trade-off.
- Typical of problems modeled by bandits (digital advertising).
  $\rightarrow$ Bayesian bandit model.

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- But, often "solved" with heuristics (no control).

What about finance?

Portfolio management with uncertain drift (Karatzas and Zhao).

The classical Merton's problem with learning
Martingale methods vs. PDE

Problem with 2 assets

2 assets

- Risk-free asset. Interest rate r.
- Risky asset:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad \sigma > 0,$$

with μ unknown.
Prior distribution on μ: $\mathrm{mes}(d\mu)$.

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Prior distribution on μ: $\mathrm{mes}(d\mu)$.

Portfolio dynamics

$$\begin{aligned} dV_t &= ((\mu - r)\,\theta_t V_t + rV_t)\,dt + \sigma\theta_t V_t dW_t \\ &= ((\mu - r)\,M_t + rV_t)\,dt + \sigma M_t dW_t \end{aligned}$$

Problem with 2 assets

Objective function

$$\sup_{\theta \in \mathcal{A}} \mathbb{E}\left[U\left(V_T^{0,V_0,\theta}\right)\right],$$

where $\mathcal{A}$ is the set of admissible strategies (see paper).

Strategies must be adapted to $\mathcal{F}^S$.

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Strategies must be adapted to $\mathcal{F}^S$.

Two approaches

- Karatzas and Zhao: martingale method (article from 98, not much known)
- Guéant and Pu: PDE method with conjugate priors. Can be generalized to non-martingale frameworks.

Martingale method

Introduction of martingale measure $\mathbb{Q}$

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T = e^{-\frac{\mu-r}{\sigma}W_T - \frac{1}{2\sigma^2}(\mu-r)^2 T},$$

$$W_t^{\mathbb{Q}} = W_t + \frac{\mu-r}{\sigma},$$

such that

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}. \\ dV_t &= rV_t + \sigma\theta V_t dW_t^{\mathbb{Q}}. \end{aligned}$$

Warning: Z_T is not $\mathcal{F}_T^S$-measurable. But $W^{\mathbb{Q}}$ is $\mathcal{F}^S$-adapted.

Martingale method

Concavity of U

$$\begin{aligned}\mathbb{E}\left[U\left(V_T^{0,V_0,\theta}\right)\right] &\leq \mathbb{E}\left[U\left(V_T^{0,V_0,\theta^\star}\right)\right] \\ &+\mathbb{E}\left[U'\left(V_T^{0,V_0,\theta^\star}\right)(V_T^{0,V_0,\theta}-V_T^{0,V_0,\theta^\star})\right]\end{aligned}$$

Martingale method

Concavity of U

$$
\begin{aligned}
\mathbb{E}\left[U\left(V_T^{0,V_0,\theta}\right)\right] &\leq \mathbb{E}\left[U\left(V_T^{0,V_0,\theta^\star}\right)\right] \\
&\quad + \mathbb{E}\left[U'\left(V_T^{0,V_0,\theta^\star}\right)\left(V_T^{0,V_0,\theta} - V_T^{0,V_0,\theta^\star}\right)\right] \\
&\leq \mathbb{E}\left[U\left(V_T^{0,V_0,\theta^\star}\right)\right] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{Z_T} U'\left(V_T^{0,V_0,\theta^\star}\right)\left(V_T^{0,V_0,\theta} - V_T^{0,V_0,\theta^\star}\right)\right]
\end{aligned}
$$

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&\leq \mathbb{E}\left[U\left(V_T^{0,V_0,\theta^\star}\right)\right] + \\
&\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[1/Z_T|\mathcal{F}_T^S\right]U'\left(V_T^{0,V_0,\theta^\star}\right)\left(V_T^{0,V_0,\theta} - V_T^{0,V_0,\theta^\star}\right)\right].
\end{aligned}
$$

Martingale method

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\mathbb{E}\left[U\left(V_T^{0,V_0,\theta}\right)\right] &\leq \mathbb{E}\left[U\left(V_T^{0,V_0,\theta^\star}\right)\right] \\
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&\leq \mathbb{E}\left[U\left(V_T^{0,V_0,\theta^\star}\right)\right] \\
&\quad +\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{Z_T}U'\left(V_T^{0,V_0,\theta^\star}\right)\left(V_T^{0,V_0,\theta} - V_T^{0,V_0,\theta^\star}\right)\right] \\
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&\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[1/Z_T|\mathcal{F}_T^S\right]U'\left(V_T^{0,V_0,\theta^\star}\right)\left(V_T^{0,V_0,\theta} - V_T^{0,V_0,\theta^\star}\right)\right].
\end{aligned}
$$

If $U'(V_T^{0,V_0,\theta^\star}) = \frac{ce^{-rT}}{\mathbb{E}^{\mathbb{Q}}[1/Z_T|\mathcal{F}_T^S]}$, then $\theta^\star$ is optimal!

Karatzas and Zhao results

Next steps

- $\mathbb{E}^{\mathbb{Q}}\left[1/Z_T | \mathcal{F}_T^S\right]$ must be computed:

$$\int_{\mathbb{R}} e^{\frac{x-r}{\sigma} W_T^{\mathbb{Q}} - \frac{1}{2\sigma^2}(x-r)^2 T} \mathrm{mes}(dx).$$

- Identification of c as above.
- Identification of $\theta^\star$ as above.

Karatzas and Zhao results

Next steps

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- Identification of c as above.
- Identification of $\theta^\star$ as above.

Advantages and drawbacks

- $\mathrm{mes}(d\mu)$ can be very general.
- U is general.
- (Very) painful computations.
- Requires martingales.

Bayesian learning

We consider a conjugate (Gaussian) prior for μ:

Bayesian prior on μ

$$\mu \sim \mathcal{N}(\beta_0, \nu_0^2)$$

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Observing the evolution of S enables to update the prior belief.

Dynamics of the beliefs

$$\mu \sim \mathcal{N}(\beta_t, \nu_t^2)$$

and Bayes' rule gives:

$$\nu_t^2 = \frac{\sigma^2 \nu_0^2}{\sigma^2 + \nu_0^2 t}$$

$$d\beta_t = g(t)\left(\frac{dS_t}{S_t} - \beta_t dt\right), \qquad g(t) = \frac{\nu_0^2}{\sigma^2 + \nu_0^2 t}.$$

Portfolio dynamics

We introduce a new ($\mathcal{F}^S$-adapted) Brownian motion:

$$\widehat{W}_t = W_t + \int_0^t \frac{\mu - \beta_s}{\sigma} ds.$$

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Dynamics of state variables

$$\begin{aligned} dV_t &= ((\beta_t - r)\,\theta_t V_t + rV_t)\,dt + \sigma\theta_t V_t d\widehat{W}_t \\ &= ((\beta_t - r)\,M_t + rV_t)\,dt + \sigma M_t d\widehat{W}_t. \\ d\beta_t &= \sigma g(t) d\widehat{W}_t. \end{aligned}$$

Portfolio dynamics

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&= ((\beta_t - r)\, M_t + rV_t)\, dt + \sigma M_t d\widehat{W}_t. \\
d\beta_t &= \sigma g(t) d\widehat{W}_t.
\end{aligned}$$

$\rightarrow \beta$ **is a new state variable.**

Value function and HJB equation

Value function

$$v(t, V, \beta) = \sup_{\theta \in \mathcal{A}_t} \mathbb{E}\left[U\left(V_T^{t,V,\beta,\theta}\right)\right]$$

Value function and HJB equation

Value function

$$v(t, V, \beta) = \sup_{\theta \in \mathcal{A}_t} \mathbb{E}\left[U\left(V_T^{t,V,\beta,\theta}\right)\right]$$

HJB equation

$$-\partial_t u(t, V, \beta) - \frac{1}{2}\sigma^2 g^2(t)\, \partial^2_{\beta\beta} u(t, V, \beta)$$
$$-\sup_{\theta} \left\{ ((\beta - r)\theta + r)\, V \partial_V u(t, V, \beta) \right.$$
$$\left. +\frac{1}{2}\sigma^2 \theta^2 V^2 \partial^2_{VV} u(t, V, \beta) + \sigma^2 g(t)\, \theta V \partial^2_{V\beta} u(t, V, \beta) \right\} = 0,$$

with terminal condition

$$u(T, V, \beta) = U(V).$$

Solution in the CARA case

Ansatz

$$u(t, V, \beta) = -\exp\left[-\gamma\left(e^{r(T-t)}V + \varphi(t, \beta)\right)\right].$$

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$$u(t, V, \beta) = -\exp\left[-\gamma\left(e^{r(T-t)}V + \varphi(t, \beta)\right)\right].$$

Equation for φ: a linear PDE!

$$-\partial_t\varphi(t, \beta) - \frac{1}{2}\sigma^2 g^2(t)\,\partial^2_{\beta\beta}\varphi(t, \beta)$$

$$-\frac{(\beta - r)^2}{2\gamma\sigma^2} + g(t)(\beta - r)\,\partial_\beta\varphi(t, \beta) = 0,$$

with terminal condition

$$\varphi(T, \beta) = 0.$$

Solution in the CARA case

Optimizer

$$M^{\star} \quad = \quad e^{-r(T-t)} \left(\frac{(\beta - r)}{\gamma \sigma^2} - g\left(t\right) \partial_{\beta} \varphi\left(t, \beta\right) \right).$$

Solution in the CARA case

Optimizer

$$M^{\star} = e^{-r(T-t)}\left(\frac{(\beta - r)}{\gamma\sigma^2} - g(t)\,\partial_\beta\varphi(t,\beta)\right).$$

Solution φ

$$\varphi(t,\beta) = a(t) + \frac{1}{2}b(t)(\beta - r)^2$$

$$\begin{cases} a'(t) + \frac{1}{2}\sigma^2 g^2(t)\,b(t) = 0 \\ b'(t) + \frac{1}{\gamma\sigma^2} - 2g(t)\,b(t) = 0. \end{cases}$$

with terminal condition $a(T) = b(T) = 0$.

Solution in the CARA case

Solutions a and b

$$
\begin{aligned}
a(t) &= \frac{1}{2\gamma}\left(\log\frac{g(t)}{g(T)} - (T-t)\,g(T)\right) \\
b(t) &= \frac{1}{\gamma\sigma^2}(T-t)\,\frac{g(T)}{g(t)}
\end{aligned}
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Optimizer

$$
M_t^{\star} = e^{-r(T-t)}\frac{g(T)}{g(t)}\frac{\beta_t - r}{\gamma\sigma^2}.
$$

Solution in the CARA case

Solutions a and b

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Optimizer

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$$

The verification approach works for L^2 adapted processes M with linear growth in $\widehat{W}$.

Comments

The optimizer is

$$M_t^\star = e^{-r(T-t)} \frac{g(T)}{g(t)} \frac{\beta_t - r}{\gamma\sigma^2}.$$

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If μ was known, then

$$M_{t,\mu \text{ known}}^\star = e^{-r(T-t)} \frac{\mu - r}{\gamma\sigma^2}.$$

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$$M_t^{\star} \quad = \quad e^{-r(T-t)} \frac{g(T)}{g(t)} \frac{\beta_t - r}{\gamma\sigma^2}.$$

If μ was known, then

$$M_{t,\mu \text{ known}}^{\star} = e^{-r(T-t)} \frac{\mu - r}{\gamma\sigma^2}.$$

The naive strategy

$$M_{t,\text{naive}} = e^{-r(T-t)} \frac{\beta_t - r}{\gamma\sigma^2}$$

is suboptimal because we learn AND we know that we will learn!

Solution in the CRRA case

Ansatz

$$u\left(t, V, \beta\right) = \frac{\left(e^{r\left(T-t\right)}V\right)^{1-\gamma}}{1-\gamma}\exp\left[\varphi\left(t, \beta\right)\right].$$

Solution in the CRRA case

Ansatz

$$u(t, V, \beta) = \frac{\left(e^{r(T-t)} V\right)^{1-\gamma}}{1-\gamma} \exp\left[\varphi(t, \beta)\right].$$

Equation for φ: a nonlinear PDE

$$-\frac{1}{1-\gamma}\partial_t \varphi(t, \beta) - \frac{1}{2(1-\gamma)}\sigma^2 g^2(t)\, \partial^2_{\beta\beta}\varphi(t, \beta)$$

$$-\frac{1}{2\gamma(1-\gamma)}\sigma^2 g^2(t)\left(\partial_\beta \varphi(t, \beta)\right)^2 - \frac{1}{\gamma}\frac{(\beta - r)^2}{2\sigma^2} - \frac{1}{\gamma} g(t)(\beta - r)\,\partial_\beta \varphi(t, \beta) = 0,$$

with terminal condition

$$\varphi(T, \beta) = 0.$$

Solution in the CRRA case

Optimizer

$$\theta^{\star} = \frac{\beta - r}{\gamma\sigma^{2}} + \frac{1}{\gamma} g\left(t\right) \partial_{\beta}\varphi\left(t, \beta\right).$$

Solution in the CRRA case

Optimizer

$$\theta^\star = \frac{\beta - r}{\gamma\sigma^2} + \frac{1}{\gamma} g(t)\, \partial_\beta \varphi(t, \beta).$$

Solution φ

$$\varphi(t, \beta) = a(t) + \frac{1}{2} b(t) (\beta - r)^2$$

$$\begin{cases} a'(t) + \frac{1}{2}\sigma^2 g^2(t)\, b(t) = 0 \\ b'(t) + \frac{1}{\gamma}\sigma^2 g^2(t)\, b^2(t) + \frac{1-\gamma}{\gamma}\frac{1}{\sigma^2} + 2\frac{1-\gamma}{\gamma} g(t)\, b(t) = 0, \end{cases}$$

with terminal condition $a(T) = b(T) = 0$.

Solution in the CRRA case

Solutions a and b

$$\begin{aligned} a(t) &= \frac{\gamma}{2}\log\frac{\gamma g(t)}{(\gamma-1)g(t)+g(T)} + \frac{1}{2}\log\frac{g(T)}{g(t)} \\ b(t) &= \frac{(1-\gamma)}{\sigma^2}\frac{1}{g(t)}\frac{g(t)-g(T)}{(\gamma-1)g(t)+g(T)}. \end{aligned}$$

Solution in the CRRA case

Solutions a and b

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\begin{aligned}
a(t) &= \frac{\gamma}{2}\log\frac{\gamma g(t)}{(\gamma-1)\,g(t)+g(T)}+\frac{1}{2}\log\frac{g(T)}{g(t)} \\
b(t) &= \frac{(1-\gamma)}{\sigma^2}\frac{1}{g(t)}\frac{g(t)-g(T)}{(\gamma-1)\,g(t)+g(T)}.
\end{aligned}
$$

The solution is defined on $[0,T]$ if $\gamma \geq 1$ but there is a blow up in finite time if $\gamma < 1$.

Optimizer in the CRRA case

Optimizer

$$\theta_t^\star = \frac{\beta_t - r}{\gamma\sigma^2} \frac{\gamma g(T)}{(\gamma - 1) g(t) + g(T)}.$$

Optimizer in the CRRA case

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$$\theta_t^\star \quad = \quad \frac{\beta_t - r}{\gamma\sigma^2} \frac{\gamma g\left(T\right)}{\left(\gamma - 1\right) g\left(t\right) + g\left(T\right)}.$$

- If $\gamma > 1$, then the learning-anticipation effect is the same as in the CARA case.

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- For $\gamma = 1$, there is no learning-anticipation effect.

Optimizer in the CRRA case

Optimizer

$$\theta_t^\star = \frac{\beta_t - r}{\gamma\sigma^2}\frac{\gamma g(T)}{(\gamma-1)g(t)+g(T)}.$$

- If $\gamma > 1$, then the learning-anticipation effect is the same as in the CARA case.
- For $\gamma = 1$, there is no learning-anticipation effect.
- If $\gamma < 1$, the effect is more complex, because there is a blow up.

Remarks

- All the formulas can be extended to the case of $d > 1$ risky assets (see next slide).
- Two important ideas:
  - Extension of the state space (not always necessary).
  - Markovian dynamics thanks to conjugate priors.
- The PDE method can be used in many models.

Multi-asset extension

Main changes:

- σ is replaced by a covariance matrix Σ.
- $\mu \sim \mathcal{N}(\beta_0, \Gamma_0)$.

Bayes' rule gives:

$$\Gamma_t = \left(\Gamma_0^{-1} + t\Sigma^{-1}\right)^{-1}$$

$$d\beta_t = \Gamma_t \Sigma^{-1} (\mu - \beta_t)\, dt + \Gamma_t \Sigma^{-1} (\sigma \odot dW_t)$$

Multi-asset extension

Optimum

- CARA case:

$$M^\star \quad = \quad e^{-r(T-t)}\frac{1}{\gamma}\Sigma^{-1}\Gamma_T\Gamma_t^{-1}\left(\beta - r\vec{1}\right).$$

- CRRA case:

$$\theta^\star \quad = \quad \Sigma^{-1}\left(\Gamma_t^{-1} + (\gamma - 1)\,\Gamma_T^{-1}\right)^{-1}\Gamma_t^{-1}\left(\beta - r\vec{1}\right).$$

What about the Almgren-Chriss framework?

Mixing Almgren-Chriss and Merton (with learning)

Almgren-Chriss framework

- Time: t.
- Number of shares: $q_t = q_0 + \int_0^t v_s ds$.
- Price: $dS_t = \mu dt + \sigma dW_t$, $\underline{\mu \text{ unknown}}$.
- Cash: $dX_t = -v_t S_t dt - V_t L\left(\frac{v_t}{V_t}\right) dt, \qquad X_0 = 0.$

Mixing Almgren-Chriss and Merton (with learning)

Almgren-Chriss framework

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Optimization problem

$$\sup_{(v_t)_t \in \mathcal{A}} \mathbb{E}\left[-\exp(-\gamma(X_T + q_T S_T - \ell(q_T)))\right], \qquad T \text{ fixed}$$

Bayesian learning

Bayesian prior on μ

$$\mu \sim \mathcal{N}(\beta_0, \nu_0)$$

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Observing the evolution of S enables to update the prior belief.

Dynamics of the beliefs

$$\mu \sim \mathcal{N}(\beta_t, \nu_t)$$

and Bayes' rule gives:

$$\nu_t^2 = \frac{\sigma^2 \nu_0^2}{\sigma^2 + \nu_0^2 t}$$

$$d\beta_t = g(t)(dS_t - \beta_t dt), \qquad g(t) = \frac{\nu_0^2}{\sigma^2 + \nu_0^2 t}$$

A new Brownian motion

Brownian motion adapted to the filtration of observables

$$\widehat{W}_t = W_t + \int_0^t \frac{\mu - \beta_s}{\sigma} ds$$

A new Brownian motion

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Dynamics of the state variables

- Number of shares: $q_t = q_0 + \int_0^t v_s ds$
- Price: $dS_t = \beta_t dt + \sigma d\widehat{W}_t$
- Cash: $dX_t = -v_t S_t dt - V_t L\left(\frac{v_t}{V_t}\right) dt$
- Belief: $d\beta_t = \sigma g(t) d\widehat{W}_t$

HJB Equation

The HJB equation associated with the extended stochastic optimal control problem is:

HJB equation

$$0 = \partial_t u + \beta \partial_S u + \sup_{v \in \mathbb{R}} \left\{ v \partial_q u + \left(-vS - L\left(\frac{v}{V_t}\right) V_t \right) \partial_x u \right\}$$

$$+ \frac{1}{2}\sigma^2 \partial^2_{SS} u + \frac{1}{2}\sigma^2 g(t)^2 \partial^2_{\beta\beta} u + \sigma^2 g(t) \partial^2_{\beta S} u$$

with terminal condition:

$$u(T, x, q, S, \beta) \quad = \quad -\exp(-\gamma(x + qS - \ell(q)))$$

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HJB equation

$$0 = \partial_t u + \beta \partial_S u + \sup_{v \in \mathbb{R}} \left\{ v \partial_q u + \left(-vS - L\left(\frac{v}{V_t}\right) V_t \right) \partial_x u \right\}$$

$$+ \frac{1}{2}\sigma^2 \partial^2_{SS} u + \frac{1}{2}\sigma^2 g(t)^2 \partial^2_{\beta\beta} u + \sigma^2 g(t) \partial^2_{\beta S} u$$

with terminal condition:

$$u(T, x, q, S, \beta) \quad = \quad -\exp(-\gamma(x + qS - \ell(q)))$$

We control and we learn, but we control knowing that we shall continue to learn.

Change of variables

We use the following ansatz:

Definition

We introduce θ by:

$$u(t,x,q,S) = -\exp\left(-\gamma\left(x+qS-\theta(t,q,\beta)\right)\right)$$

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PDE

$$0 = \partial_t \theta - \beta q + \frac{1}{2}\gamma\sigma^2 q^2 - V_t H(\partial_q \theta)$$

$$+\frac{1}{2}\sigma^2 g(t)^2 (\partial^2_{\beta\beta}\theta + \gamma(\partial_\beta \theta)^2) - \gamma\sigma^2 g(t) q \partial_\beta \theta$$

with $\theta(T, q, \beta) = \ell(q)$.

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with $\theta(T,q,\beta) = \ell(q)$.

$$v^\star(t,q,\beta) = -V_t H'(\partial_q\theta(t,q,\beta)).$$

Quadratic case – Portfolio choice

If $L(\rho) = \eta\rho^2$ and $\ell(q) = \frac{1}{2}Kq^2$, then a natural ansatz is

$$\theta(t, q, \beta) \;\; = \;\; a(t) + \frac{1}{2}b(t)\beta^2 + c(t)\beta q + \frac{1}{2}d(t)q^2$$

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The PDE boils down to a system of ODEs:

ODEs

$$\begin{aligned}
a' &= -\frac{1}{2}\sigma^2 g^2 b, \qquad a(T) = 0 \\
b' &= -\gamma\sigma^2 g^2 b^2 + \frac{V}{2\eta}c^2, \qquad b(T) = 0 \\
c' &= 1 - \gamma\sigma^2 g^2 bc + \gamma\sigma^2 gb + \frac{V}{2\eta}cd, \qquad c(T) = 0 \\
d' &= -\gamma\sigma^2 - \gamma\sigma^2 g^2 c^2 + 2\gamma\sigma^2 gc + \frac{V}{2\eta}d^2, \qquad d(T) = K
\end{aligned}$$

Quadratic case – Portfolio transition (relaxed)

If $L(\rho) = \eta\rho^2$ and $\ell(q) = \frac{1}{2}K(q - q_{\text{target}})^2$, then a natural ansatz is

$$\theta(t, q, \beta) \quad = \quad a(t) + \frac{1}{2}b(t)\beta^2 + c(t)\beta q + \frac{1}{2}d(t)q^2 + e(t)\beta + f(t)q$$

Quadratic case – Portfolio transition (relaxed)

If $L(\rho) = \eta\rho^2$ and $\ell(q) = \frac{1}{2}K(q - q_{\text{target}})^2$, then a natural ansatz is

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The PDE boils down to a system of ODEs

$$\begin{aligned}
a' &= -\frac{1}{2}\sigma^2 g^2 b - \frac{1}{2}\gamma\sigma^2 g^2 e^2 + \frac{V}{4\eta}f^2, \qquad a(T) = \frac{1}{2}Kq_{\text{target}}^2 \\
b' &= -\gamma\sigma^2 g^2 b^2 + \frac{V}{2\eta}c^2, \qquad b(T) = 0 \\
c' &= 1 - \gamma\sigma^2 g^2 bc + \gamma\sigma^2 gb + \frac{V}{2\eta}cd, \qquad c(T) = 0 \\
d' &= -\gamma\sigma^2 - \gamma\sigma^2 g^2 c^2 + 2\gamma\sigma^2 gc + \frac{V}{2\eta}d^2, \qquad d(T) = K \\
e' &= -\gamma\sigma^2 g^2 be + \frac{V}{2\eta}cf, \qquad e(T) = 0 \\
f' &= -\gamma\sigma^2 g^2 ce + \gamma\sigma^2 ge + \frac{V}{2\eta}df, \qquad f(T) = -Kq_{\text{target}}
\end{aligned}$$

Examples

- $S_0 = 50$ €
- $\mu = 0.01$ €·day^{-1}.
- $\sigma = 0.6$ €·day$^{-1/2}$.
- $T = 10$ days.
- $V = 4\ 000\ 000$ shares·day^{-1}.
- $L(\rho) = \eta|\rho|^2$ with $\eta = 0.15$ € ·shares^{-1} · day^{-1}.
- $\gamma = 2 \cdot 10^{-7}$€$^{-1}$.
- $\beta_0 = 0.01$ €·day^{-1}.
- $\nu_0 = 0.03$ €·day^{-1}.

Examples

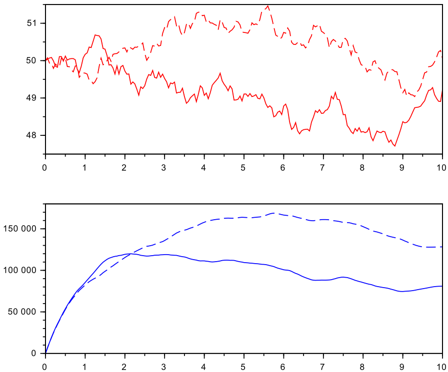

Figure: Optimal strategies for $\ell(q) = 0$.

A way to do trend following!

Concluding remarks

Control and learning

- Learning taken into account by a new state variable (not really new, because we can take S).
- Different from plugging recently estimated values (we know that we will learn).
- Less powerful than martingale methods (Karatzas-Zhao) but larger scope for applications (Almgren-Chriss).
- Many applications outside of Finance.
  $\rightarrow$ Main ingredient: conjugate distributions!

End of Lecture 3

Thank you. Questions?