Continuous-time optimal control on discrete spaces. Applications to inventory management in commerce and finance

Pr. Olivier Guéant (Université Paris 1 Panthéon-Sorbonne and ENSAE)

Spring 2021

Introduction

The lecturer

The lecturer

- Undergraduate and graduate studies: Mathematics / Computer Science / Economics (Ecole Normale Supérieure, Paris + ENSAE, Paris, + Harvard Univ.)

The lecturer

- Undergraduate and graduate studies: Mathematics / Computer Science / Economics (Ecole Normale Supérieure, Paris + ENSAE, Paris, + Harvard Univ.)
- PhD (Université Paris Dauphine) on Mean Field Games.

The lecturer

- Undergraduate and graduate studies: Mathematics / Computer Science / Economics (Ecole Normale Supérieure, Paris + ENSAE, Paris, + Harvard Univ.)
- PhD (Université Paris Dauphine) on Mean Field Games.
- First jobs in banks and in the start up I created with my PhD advisors.

The lecturer

- Academic career: Assistant Professor at Univ. Paris 7 in Applied Mathematics (Numerical analysis).

The lecturer

- Academic career: Assistant Professor at Univ. Paris 7 in Applied Mathematics (Numerical analysis).
- Current position: Full Professor of Applied Mathematics at Université Paris 1 Panthéon Sorbonne and Adjunct Professor of Finance at ENSAE.

The lecturer

- Academic career: Assistant Professor at Univ. Paris 7 in Applied Mathematics (Numerical analysis).
- Current position: Full Professor of Applied Mathematics at Université Paris 1 Panthéon Sorbonne and Adjunct Professor of Finance at ENSAE.
- Research: initially in mean field games, then in Quantitative Finance.

Greatest common divisor: optimal control theory

The lectures

Optimal control theory

The lectures

Optimal control theory

- A theory to tackle dynamic optimization problems.

The lectures

Optimal control theory

- A theory to tackle dynamic optimization problems.
- Linked to the calculus of variations (18th century) but a major achievement of the 20th century (Bellman equations, viscosity solutions, etc.).

The lectures

Optimal control theory

- A theory to tackle dynamic optimization problems.
- Linked to the calculus of variations (18th century) but a major achievement of the 20th century (Bellman equations, viscosity solutions, etc.).
- Used in a lot of fields: aerospace, robotics, finance, etc.

The lectures

Optimal control theory

- A theory to tackle dynamic optimization problems.
- Linked to the calculus of variations (18th century) but a major achievement of the 20th century (Bellman equations, viscosity solutions, etc.).
- Used in a lot of fields: aerospace, robotics, finance, etc.
- Very hot recently: related to reinforcement learning (see DeepMind).

The lectures

Optimal control theory

- A theory to tackle dynamic optimization problems.
- Linked to the calculus of variations (18th century) but a major achievement of the 20th century (Bellman equations, viscosity solutions, etc.).
- Used in a lot of fields: aerospace, robotics, finance, etc.
- Very hot recently: related to reinforcement learning (see DeepMind).

Different frameworks

The lectures

Optimal control theory

- A theory to tackle dynamic optimization problems.
- Linked to the calculus of variations (18th century) but a major achievement of the 20th century (Bellman equations, viscosity solutions, etc.).
- Used in a lot of fields: aerospace, robotics, finance, etc.
- Very hot recently: related to reinforcement learning (see DeepMind).

Different frameworks

- Discrete-time with discrete/continuous-state space: recursive equations (often untractable).

The lectures

Optimal control theory

- A theory to tackle dynamic optimization problems.
- Linked to the calculus of variations (18th century) but a major achievement of the 20th century (Bellman equations, viscosity solutions, etc.).
- Used in a lot of fields: aerospace, robotics, finance, etc.
- Very hot recently: related to reinforcement learning (see DeepMind).

Different frameworks

- Discrete-time with discrete/continuous-state space: recursive equations (often untractable).
- Continuous-time with continuous state space: partial differential equations (sometimes very technical, e.g. viscosity solutions).

The lectures

Optimal control theory

- A theory to tackle dynamic optimization problems.
- Linked to the calculus of variations (18th century) but a major achievement of the 20th century (Bellman equations, viscosity solutions, etc.).
- Used in a lot of fields: aerospace, robotics, finance, etc.
- Very hot recently: related to reinforcement learning (see DeepMind).

Different frameworks

- Discrete-time with discrete/continuous-state space: recursive equations (often untractable).
- Continuous-time with continuous state space: partial differential equations (sometimes very technical, e.g. viscosity solutions).
- **Continuous-time with discrete state space: ordinary differential equations (less technical, and reveals the main ideas)**.

The lectures

In this lecture

The lectures

In this lecture

- Introduction of the modelling framework and presentation of the main issues.

The lectures

In this lecture

- Introduction of the modelling framework and presentation of the main issues.
- Motivation with a toy example from (re)commerce.

The lectures

In this lecture

- Introduction of the modelling framework and presentation of the main issues.
- Motivation with a toy example from (re)commerce.
- Derivation of the main results.

The lectures

In this lecture

- Introduction of the modelling framework and presentation of the main issues.
- Motivation with a toy example from (re)commerce.
- Derivation of the main results.

In the next lecture

- Derivation of the main results (continued).

The lectures

In this lecture

- Introduction of the modelling framework and presentation of the main issues.
- Motivation with a toy example from (re)commerce.
- Derivation of the main results.

In the next lecture

- Derivation of the main results (continued).
- The specific case of entropic costs.

The lectures

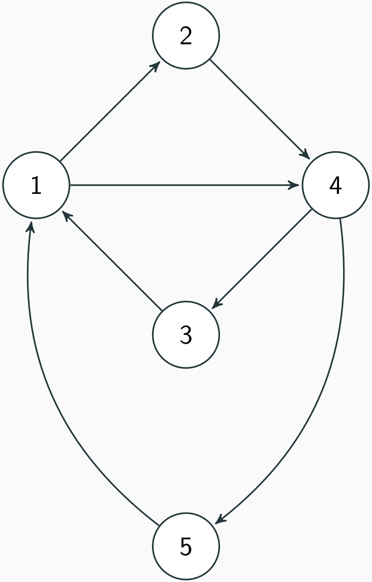
In this lecture

- Introduction of the modelling framework and presentation of the main issues.
- Motivation with a toy example from (re)commerce.
- Derivation of the main results.

In the next lecture

- Derivation of the main results (continued).
- The specific case of entropic costs.
- Discussion of applications to market making issues.

Introduction to the modelling framework: graphs



Introduction to the modelling framework: graphs

Vocabulary

Introduction to the modelling framework: graphs

Vocabulary

- Nodes or vertices: $\mathcal{I} = \{1, \dots, N\}$.

Introduction to the modelling framework: graphs

Vocabulary

- Nodes or vertices: $\mathcal{I} = \{1, \ldots, N\}$.
- Edges (directed edges) or links: for each $i \in \mathcal{I}, \mathcal{V}(i) \subset \mathcal{I} \setminus \{i\}$ is the set of nodes j for which a directed edge exists from i to j.

Introduction to the modelling framework: graphs

Vocabulary

- Nodes or vertices: $\mathcal{I} = \{1, \ldots, N\}$.
- Edges (directed edges) or links: for each $i \in \mathcal{I}, \mathcal{V}(i) \subset \mathcal{I} \setminus \{i\}$ is the set of nodes j for which a directed edge exists from i to j.
- Transition probabilities in continuous time are described by a collection of feedback control functions $(\lambda_t(i, \cdot))_{i \in \mathcal{I}}$ where $\lambda_t(i, \cdot) : \mathcal{V}(i) \to \mathbb{R}_+$.

Introduction to the modelling framework: graphs

Vocabulary

- Nodes or vertices: $\mathcal{I} = \{1, \ldots, N\}$.
- Edges (directed edges) or links: for each $i \in \mathcal{I}, \mathcal{V}(i) \subset \mathcal{I} \setminus \{i\}$ is the set of nodes j for which a directed edge exists from i to j.
- Transition probabilities in continuous time are described by a collection of feedback control functions $(\lambda_t(i, \cdot))_{i \in \mathcal{I}}$ where $\lambda_t(i, \cdot) : \mathcal{V}(i) \to \mathbb{R}_+$.

Main assumptions

- On the graph: it is connected, i.e. there is a path from any point to any other point.

Introduction to the modelling framework: graphs

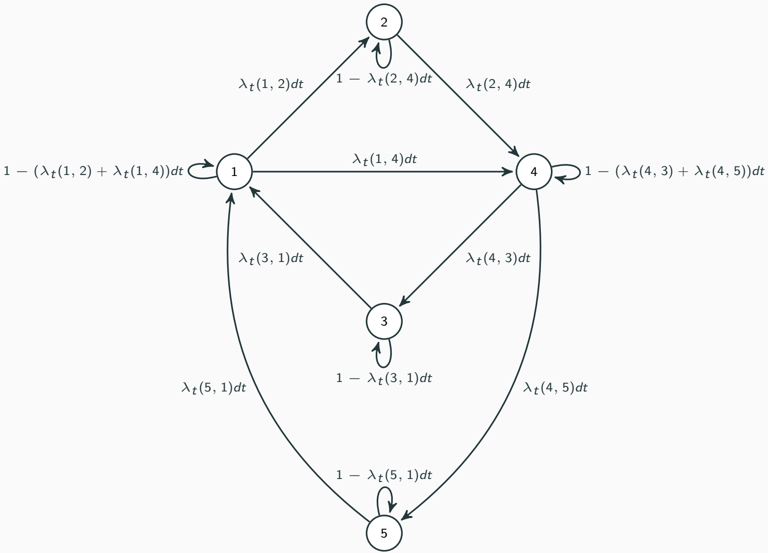
Vocabulary

- Nodes or vertices: $\mathcal{I} = \{1, \dots, N\}$.
- Edges (directed edges) or links: for each $i \in \mathcal{I}, \mathcal{V}(i) \subset \mathcal{I} \setminus \{i\}$ is the set of nodes j for which a directed edge exists from i to j.
- Transition probabilities in continuous time are described by a collection of feedback control functions $(\lambda_t(i, \cdot))_{i \in \mathcal{I}}$ where $\lambda_t(i, \cdot) : \mathcal{V}(i) \to \mathbb{R}_+$.

Main assumptions

- On the graph: it is connected, i.e. there is a path from any point to any other point.
- On transition probabilities: they are chosen by an agent. He/she cannot create edges.

Introduction to the modelling framework: graphs



Introduction to the optimization problem

An agent moving on the graph

Introduction to the optimization problem

An agent moving on the graph

- Time interval: $[0, T]$

Introduction to the optimization problem

An agent moving on the graph

- Time interval: $[0, T]$
- If at time t the agent is at node/state i, then, over $[t, t+dt]$:
  - he/she gets a payoff $h(i)dt$
  - he/she pays a cost $c\left(i, (\lambda_t(i,j))_{j\in\mathcal{V}(i)}\right) dt$

Introduction to the optimization problem

An agent moving on the graph

- Time interval: $[0, T]$
- If at time t the agent is at node/state i, then, over $[t, t+dt]$:
    - he/she gets a payoff $h(i)dt$
    - he/she pays a cost $c\left(i, (\lambda_t(i,j))_{j\in\mathcal{V}(i)}\right) dt$

$$\Rightarrow L\left(i, \left(\lambda_t\left(i,j\right)\right)_{j\in\mathcal{V}(i)}\right) = c\left(i, (\lambda_t(i,j))_{j\in\mathcal{V}(i)}\right) - h(i).$$

Introduction to the optimization problem

An agent moving on the graph

- Time interval: $[0, T]$
- If at time t the agent is at node/state i, then, over $[t, t + dt]$:
  - he/she gets a payoff $h(i)dt$
  - he/she pays a cost $c\left(i, (\lambda_t(i,j))_{j \in \mathcal{V}(i)}\right) dt$

$\Rightarrow L\left(i, (\lambda_t(i,j))_{j \in \mathcal{V}(i)}\right) = c\left(i, (\lambda_t(i,j))_{j \in \mathcal{V}(i)}\right) - h(i).$

Remark: L can take the value $+\infty$.

Introduction to the optimization problem

An agent moving on the graph

- Time interval: $[0, T]$
- If at time t the agent is at node/state i, then, over $[t, t+dt]$:
  - he/she gets a payoff $h(i)dt$
  - he/she pays a cost $c\left(i, (\lambda_t(i,j))_{j\in\mathcal{V}(i)}\right) dt$

  $\Rightarrow L\left(i, (\lambda_t(i,j))_{j\in\mathcal{V}(i)}\right) = c\left(i, (\lambda_t(i,j))_{j\in\mathcal{V}(i)}\right) - h(i).$
  Remark: L can take the value $+\infty$.
- If at time T the agent is at node/state i: final payoff $g(i)$

Introduction to the optimization problem

An agent moving on the graph

- Time interval: $[0, T]$
- If at time t the agent is at node/state i, then, over $[t, t+dt]$:
  - he/she gets a payoff $h(i)dt$
  - he/she pays a cost $c\left(i, (\lambda_t(i,j))_{j\in\mathcal{V}(i)}\right) dt$

  $\Rightarrow L\left(i, (\lambda_t(i,j))_{j\in\mathcal{V}(i)}\right) = c\left(i, (\lambda_t(i,j))_{j\in\mathcal{V}(i)}\right) - h(i).$
  Remark: L can take the value $+\infty$.
- If at time T the agent is at node/state i: final payoff $g(i)$
- Discount rate $r \geq 0$.

Introduction to the optimization problem

State process

$(X_s^{t,i,\lambda})_{s\in[t,T]}$: continuous-time Markov chain on the graph starting from node i at time t, with instantaneous transition probabilities given by λ.

Introduction to the optimization problem

State process

$(X_s^{t,i,\lambda})_{s\in[t,T]}$: continuous-time Markov chain on the graph starting from node i at time t, with instantaneous transition probabilities given by λ.

Goal of the agent

Maximizing over the intensities the objective criterion

$$\mathbb{E}\left[-\int_0^T e^{-rt} L\left(X_t^{0,i,\lambda}, \left(\lambda_t\left(X_t^{0,i,\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda}\right)}\right) dt \right.$$
$$\left. + e^{-rT} g\left(X_T^{0,i,\lambda}\right)\right]$$

Introduction to the optimization problem

State process

$(X_s^{t,i,\lambda})_{s\in[t,T]}$: continuous-time Markov chain on the graph starting from node i at time t, with instantaneous transition probabilities given by λ.

Goal of the agent

Maximizing over the intensities the objective criterion

$$\mathbb{E}\left[-\int_0^T e^{-rt} L\left(X_t^{0,i,\lambda}, \left(\lambda_t\left(X_t^{0,i,\lambda}, j\right)\right)_{j\in\mathcal{V}(X_t^{0,i,\lambda})}\right) dt + e^{-rT} g\left(X_T^{0,i,\lambda}\right)\right]$$

Remark: To be rigorous, we impose λ such that $t \mapsto \lambda_t(i,j) \in L^1(0,T)$.

Main mathematical problems

Main mathematical problems

Optimal controls

Main mathematical problems

Optimal controls

- Under what conditions do there exist optimal controls / optimal intensities?

Main mathematical problems

Optimal controls

- Under what conditions do there exist optimal controls / optimal intensities?
- How do you compute them if they exist?

Main mathematical problems

Optimal controls

- Under what conditions do there exist optimal controls / optimal intensities?
- How do you compute them if they exist?

Asymptotics

Main mathematical problems

Optimal controls

- Under what conditions do there exist optimal controls / optimal intensities?
- How do you compute them if they exist?

Asymptotics

- What happens when $T \to \infty$ if $r > 0$?

Main mathematical problems

Optimal controls

- Under what conditions do there exist optimal controls / optimal intensities?
- How do you compute them if they exist?

Asymptotics

- What happens when $T \to \infty$ if $r > 0$? $\to$ stationary problem.

Main mathematical problems

Optimal controls

- Under what conditions do there exist optimal controls / optimal intensities?
- How do you compute them if they exist?

Asymptotics

- What happens when $T \to \infty$ if $r > 0$? $\to$ stationary problem.
- What happens when $T \to \infty$ if $r = 0$?

Main mathematical problems

Optimal controls

- Under what conditions do there exist optimal controls / optimal intensities?
- How do you compute them if they exist?

Asymptotics

- What happens when $T \to \infty$ if $r > 0$? $\to$ stationary problem.
- What happens when $T \to \infty$ if $r = 0$? $\to$ ergodic problem.

Literature

Literature

On theoretical points

Literature

On theoretical points

- Guéant, Manziuk (2020). Optimal control on graphs: existence, uniqueness, and long-term behavior. ESAIM COCV.

Literature

On theoretical points

- Guéant, Manziuk (2020). Optimal control on graphs: existence, uniqueness, and long-term behavior. ESAIM COCV.
- Guéant (2021). Optimal control on finite graphs: a reference case.

Literature

On theoretical points

- Guéant, Manziuk (2020). Optimal control on graphs: existence, uniqueness, and long-term behavior. ESAIM COCV.
- Guéant (2021). Optimal control on finite graphs: a reference case.

On applications to market making

Literature

On theoretical points

- Guéant, Manziuk (2020). Optimal control on graphs: existence, uniqueness, and long-term behavior. ESAIM COCV.
- Guéant (2021). Optimal control on finite graphs: a reference case.

On applications to market making

- Guéant, Lehalle, Fernandez-Tapia (2013). Dealing with the inventory risk: a solution to the market making problem. MAFE.

Literature

On theoretical points

- Guéant, Manziuk (2020). Optimal control on graphs: existence, uniqueness, and long-term behavior. ESAIM COCV.
- Guéant (2021). Optimal control on finite graphs: a reference case.

On applications to market making

- Guéant, Lehalle, Fernandez-Tapia (2013). Dealing with the inventory risk: a solution to the market making problem. MAFE.
- Guéant (2017). Optimal market making. AMF

Literature

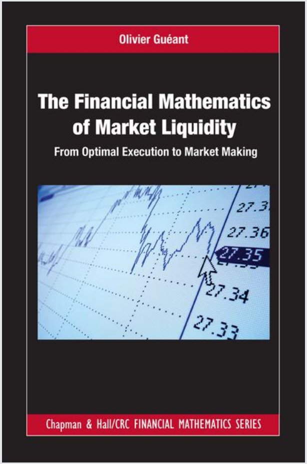
Literature

On applications to market making

Literature

On applications to market making

And of course



Motivation / Example: a toy model of commerce / recommerce

The toy problem of a platform of (re)commerce

The toy problem of a platform of (re)commerce

Buying and selling a book

- We consider a book bought and sold by a platform.

The toy problem of a platform of (re)commerce

Buying and selling a book

- We consider a book bought and sold by a platform.
- At time t, the platform proposes:

The toy problem of a platform of (re)commerce

Buying and selling a book

- We consider a book bought and sold by a platform.
- At time t, the platform proposes:
  - to buy at price $P - \delta_t^b$

The toy problem of a platform of (re)commerce

Buying and selling a book

- We consider a book bought and sold by a platform.
- At time t, the platform proposes:
  - to buy at price $P - \delta_t^b$ (if the inventory is $< Q$),

The toy problem of a platform of (re)commerce

Buying and selling a book

- We consider a book bought and sold by a platform.
- At time t, the platform proposes:
  - to buy at price $P - \delta_t^b$ (if the inventory is $< Q$),
  - to sell at price $P + \delta_t^s$

The toy problem of a platform of (re)commerce

Buying and selling a book

- We consider a book bought and sold by a platform.
- At time t, the platform proposes:
  - to buy at price $P - \delta_t^b$ (if the inventory is $< Q$),
  - to sell at price $P + \delta_t^s$ (if the inventory is > 0).

The toy problem of a platform of (re)commerce

Buying and selling a book

- We consider a book bought and sold by a platform.
- At time t, the platform proposes:
  - to buy at price $P - \delta_t^b$ (if the inventory is $< Q$),
  - to sell at price $P + \delta_t^s$ (if the inventory is > 0).
- The probability of trades over $[t, t + dt]$ are:
  - $\Lambda^b(\delta_t^b)dt$ for a buy trade (Λ^b decreasing),
  - $\Lambda^s(\delta_t^s)dt$ for a sell trade (Λ^s decreasing).

The toy problem of a platform of (re)commerce

Buying and selling a book

- We consider a book bought and sold by a platform.
- At time t, the platform proposes:
  - to buy at price $P - \delta_t^b$ (if the inventory is $< Q$),
  - to sell at price $P + \delta_t^s$ (if the inventory is > 0).
- The probability of trades over $[t, t + dt]$ are:
  - $\Lambda^b(\delta_t^b)dt$ for a buy trade (Λ^b decreasing),
  - $\Lambda^s(\delta_t^s)dt$ for a sell trade (Λ^s decreasing).
- The cost of holding an inventory q_t over $[t, t + dt]$ is $c(q_t)dt$ (where c is increasing).

The toy problem of a platform of (re)commerce

The toy problem of a platform of (re)commerce

Variables

The toy problem of a platform of (re)commerce

Variables

Denoting by N^b and N^s the point processes of "buys" and "sells" we have:

The toy problem of a platform of (re)commerce

Variables

Denoting by N^b and N^s the point processes of "buys" and "sells" we have:

- the inventory $(q_t)_t$ verifies $q_t = N_t^b - N_t^s$.

The toy problem of a platform of (re)commerce

Variables

Denoting by N^b and N^s the point processes of "buys" and "sells" we have:

- the inventory $(q_t)_t$ verifies $q_t = N_t^b - N_t^s$.
- the money on the cash account $(Z_t)_t$ verifies:

$$dZ_t = -(P - \delta_t^b)dN_t^b + (P + \delta_t^s)dN_t^s = -Pdq_t + \delta_t^b dN_t^b + \delta_t^s dN_t^s.$$

The toy problem of a platform of (re)commerce

Variables

Denoting by N^b and N^s the point processes of "buys" and "sells" we have:

- the inventory $(q_t)_t$ verifies $q_t = N_t^b - N_t^s$.
- the money on the cash account $(Z_t)_t$ verifies:

$$dZ_t = -(P - \delta_t^b)dN_t^b + (P + \delta_t^s)dN_t^s = -Pdq_t + \delta_t^b dN_t^b + \delta_t^s dN_t^s.$$

Optimization problem

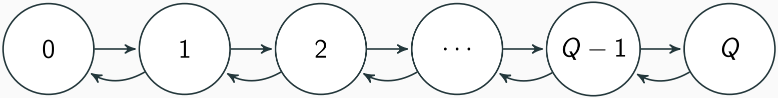
Maximizing

$$\begin{aligned} & \mathbb{E}\left[Z_T + Pq_T - \int_0^T c(q_t)dt\right] = \mathbb{E}\left[\int_0^T \delta_t^b dN_t^b + \delta_t^s dN_t^s - c(q_t)dt\right] \\ = \; & \mathbb{E}\left[\int_0^T \left(\delta_t^b \Lambda^b(\delta_t^b) + \delta_t^s \Lambda^s(\delta_t^s) - c(q_t)\right) dt\right], \qquad \lambda_t^{b/s} = \Lambda^{b/s}(\delta_t^{b/s}) \\ = \; & \mathbb{E}\left[\int_0^T \left(\left(\Lambda^b\right)^{-1}(\lambda_t^b)\lambda_t^b + (\Lambda^s)^{-1}(\lambda_t^s)\lambda_t^s - c(q_t)\right) dt\right] \end{aligned}$$

The toy problem of a platform of (re)commerce

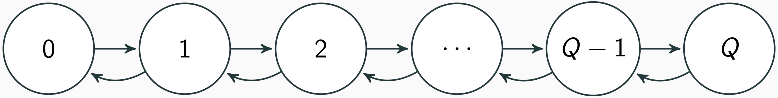
The toy problem of a platform of (re)commerce

- The graph



The toy problem of a platform of (re)commerce

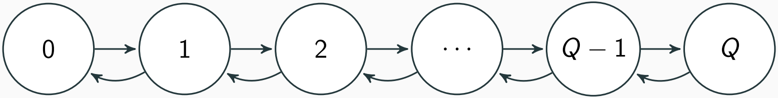
- The graph



- No discount rate.

The toy problem of a platform of (re)commerce

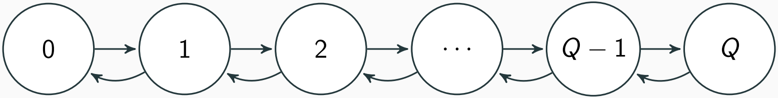
- The graph



- No discount rate.
- No final payoff.

The toy problem of a platform of (re)commerce

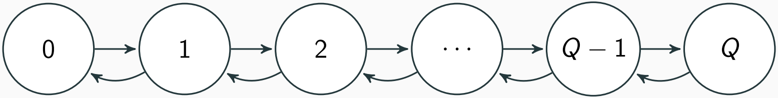
- The graph



- No discount rate.
- No final payoff.
- The function $L(\cdot, \cdot)$:

The toy problem of a platform of (re)commerce

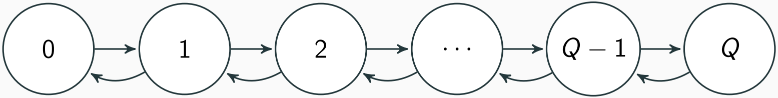
- The graph



- No discount rate.
- No final payoff.
- The function $L(\cdot, \cdot)$:
  - $L(0, \lambda(0,1)) = -\lambda(0,1) \left(\Lambda^b\right)^{-1} (\lambda(0,1)) + c(0)$

The toy problem of a platform of (re)commerce

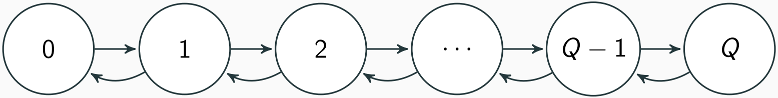
- The graph



- No discount rate.
- No final payoff.
- The function $L(\cdot,\cdot)$:
  - $L(0,\lambda(0,1)) = -\lambda(0,1)\left(\Lambda^b\right)^{-1}(\lambda(0,1)) + c(0)$
  - $L(Q,\lambda(Q,Q-1)) = -\lambda(Q,Q-1)\left(\Lambda^s\right)^{-1}(\lambda(Q,Q-1)) + c(Q)$

The toy problem of a platform of (re)commerce

- The graph



- No discount rate.
- No final payoff.
- The function $L(\cdot,\cdot)$:
  - $L(0,\lambda(0,1)) = -\lambda(0,1)\left(\Lambda^b\right)^{-1}(\lambda(0,1)) + c(0)$
  - $L(Q,\lambda(Q,Q-1)) = -\lambda(Q,Q-1)\left(\Lambda^s\right)^{-1}(\lambda(Q,Q-1)) + c(Q)$
  - $\forall q \in \{1,\dots,Q-1\}$,

$$L(q,\lambda(q,q+1),\lambda(q,q-1)) = -\lambda(q,q+1)\left(\Lambda^b\right)^{-1}(\lambda(q,q+1)) - \lambda(q,q-1)\left(\Lambda^s\right)^{-1}(\lambda(q,q-1)) + c(q)$$

A general theory for optimal control on graphs – Finite-horizon problem

Main tool of optimal control: value function

Value function

Main tool of optimal control: value function

Value function

The value function associates a state i and a time t to the best possible score starting at time t from state i:

Main tool of optimal control: value function

Value function

The value function associates a state i and a time t to the best possible score starting at time t from state i:

$$u_i^{T,r}(t) = \sup_{(\lambda_s(\cdot,\cdot))_{s\in[t,T]}} \mathbb{E}\left[-\int_t^T e^{-r(s-t)} L\left(X_s^{t,i,\lambda}, \left(\lambda_s\left(X_s^{t,i,\lambda},j\right)\right)_{j\in\mathcal{V}\left(X_s^{t,i,\lambda}\right)}\right) ds \right.$$
$$\left. + e^{-r(T-t)} g\left(X_T^{t,i,\lambda}\right)\right].$$

Main tool of optimal control: value function

Value function

The value function associates a state i and a time t to the best possible score starting at time t from state i:

$$u_i^{T,r}(t) = \sup_{(\lambda_s(\cdot,\cdot))_{s\in[t,T]}} \mathbb{E}\left[-\int_t^T e^{-r(s-t)} L\left(X_s^{t,i,\lambda}, \left(\lambda_s\left(X_s^{t,i,\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_s^{t,i,\lambda}\right)}\right) ds + e^{-r(T-t)} g\left(X_T^{t,i,\lambda}\right)\right].$$

Many methods of optimal control are based on computing the value function and deducing the optimal controls.

Main tool of optimal control: value function

Value function

The value function associates a state i and a time t to the best possible score starting at time t from state i:

$$u_i^{T,r}(t) = \sup_{(\lambda_s(\cdot,\cdot))_{s\in[t,T]}} \mathbb{E}\left[-\int_t^T e^{-r(s-t)} L\left(X_s^{t,i,\lambda}, \left(\lambda_s\left(X_s^{t,i,\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_s^{t,i,\lambda}\right)}\right) ds \right.$$
$$\left. + e^{-r(T-t)} g\left(X_T^{t,i,\lambda}\right)\right].$$

Many methods of optimal control are based on computing the value function and deducing the optimal controls.

How to compute the value function? $\rightarrow$ through the system of ODEs it solves: Hamilton-Jacobi / Bellman equations.

Heuristic derivation of Hamilton-Jacobi / Bellman equations

Heuristic derivation of Hamilton-Jacobi / Bellman equations

- Let us consider a time $t \in [0, T)$ and let us assume that we know the values of the value function at time $t + dt$.

Heuristic derivation of Hamilton-Jacobi / Bellman equations

- Let us consider a time $t \in [0, T)$ and let us assume that we know the values of the value function at time $t + dt$.
- If the agent is in state i at time t and chooses $\lambda_t(\cdot, \cdot)$ for the period $[t, t + dt]$ then:

Heuristic derivation of Hamilton-Jacobi / Bellman equations

- Let us consider a time $t \in [0, T)$ and let us assume that we know the values of the value function at time $t + dt$.
- If the agent is in state i at time t and chooses $\lambda_t(\cdot, \cdot)$ for the period $[t, t + dt]$ then:
  - for all $j \in \mathcal{V}(i)$, the agent will be in state j at time $t + dt$ with probability $\lambda_t(i, j)dt$,

Heuristic derivation of Hamilton-Jacobi / Bellman equations

- Let us consider a time $t \in [0, T)$ and let us assume that we know the values of the value function at time $t + dt$.
- If the agent is in state i at time t and chooses $\lambda_t(\cdot, \cdot)$ for the period $[t, t + dt]$ then:
  - for all $j \in \mathcal{V}(i)$, the agent will be in state j at time $t + dt$ with probability $\lambda_t(i, j)dt$,
  - the agent will still be in state i at time $t + dt$ with probability $1 - \sum_{j \in \mathcal{V}(i)} \lambda_t(i, j)dt$.

Heuristic derivation of Hamilton-Jacobi / Bellman equations

- Let us consider a time $t \in [0, T)$ and let us assume that we know the values of the value function at time $t + dt$.
- If the agent is in state i at time t and chooses $\lambda_t(\cdot,\cdot)$ for the period $[t, t+dt]$ then:
  - for all $j \in \mathcal{V}(i)$, the agent will be in state j at time $t + dt$ with probability $\lambda_t(i,j)dt$,
  - the agent will still be in state i at time $t + dt$ with probability $1 - \sum_{j\in\mathcal{V}(i)} \lambda_t(i,j)dt$.
- Therefore

$$u_i^{T,r}(t) = \sup_{\lambda_t(\cdot,\cdot)} \left\{ - L\left(i, (\lambda_t\left(i,j\right))_{j\in\mathcal{V}(i)}\right) dt + e^{-rdt} \times \right.$$
$$\left.\left(\left(1 - \sum_{j\in\mathcal{V}(i)} \lambda_t(i,j)dt\right) \cdot u_i^{T,r}(t+dt) + \sum_{j\in\mathcal{V}(i)} \lambda_t(i,j)dt \cdot u_j^{T,r}(t+dt)\right)\right\}$$

Heuristic derivation of Hamilton-Jacobi / Bellman equations

Taylor expansion

$$
\begin{aligned}
& e^{-rdt}\left(\left(1-\sum_{j\in\mathcal{V}(i)}\lambda_t(i,j)dt\right)\cdot u_i^{T,r}(t+dt)+\sum_{j\in\mathcal{V}(i)}\lambda_t(i,j)dt\cdot u_j^{T,r}(t+dt)\right)\\
=&(1-rdt)\left(u_i^{T,r}(t+dt)+\sum_{j\in\mathcal{V}(i)}\lambda_t(i,j)dt(u_j^{T,r}(t+dt)-u_i^{T,r}(t+dt))\right)\\
=&(1-rdt)\left(u_i^{T,r}(t)+\frac{d}{dt}u_i^{T,r}(t)dt+\sum_{j\in\mathcal{V}(i)}\lambda_t(i,j)dt(u_j^{T,r}(t)-u_i^{T,r}(t))+o(dt)\right)\\
=&u_i^{T,r}(t)+dt\left(-ru_i^{T,r}(t)+\frac{d}{dt}u_i^{T,r}(t)+\sum_{j\in\mathcal{V}(i)}\lambda_t(i,j)(u_j^{T,r}(t)-u_i^{T,r}(t))\right)\\
&+o(dt)
\end{aligned}
$$

Heuristic derivation of Hamilton-Jacobi / Bellman equations

Wrapping up we get:

Heuristic derivation of Hamilton-Jacobi / Bellman equations

Wrapping up we get:

$$u_i^{T,r}(t) = \sup_{\lambda_t(\cdot,\cdot)} \Bigg\{ - L\left(i, (\lambda_t(i,j))_{j\in\mathcal{V}(i)}\right) dt +$$
$$u_i^{T,r}(t) + dt \left(-r u_i^{T,r}(t) + \frac{d}{dt} u_i^{T,r}(t) + \sum_{j\in\mathcal{V}(i)} \lambda_t(i,j)(u_j^{T,r}(t) - u_i^{T,r}(t)) \right) + o(dt) \Bigg\}$$

Heuristic derivation of Hamilton-Jacobi / Bellman equations

Wrapping up we get:

$$u_i^{T,r}(t) = \sup_{\lambda_t(\cdot,\cdot)} \Bigg\{ - L\left(i, (\lambda_t(i,j))_{j\in\mathcal{V}(i)}\right) dt +$$
$$u_i^{T,r}(t) + dt\left(-ru_i^{T,r}(t) + \frac{d}{dt}u_i^{T,r}(t) + \sum_{j\in\mathcal{V}(i)} \lambda_t(i,j)(u_j^{T,r}(t) - u_i^{T,r}(t))\right) + o(dt)\Bigg\}$$

So, necessarily:

$$\begin{aligned} 0 &= \frac{d}{dt}u_i^{T,r}(t) - ru_i^{T,r}(t) \\ &\quad + \sup_{\lambda_t(\cdot,\cdot)} \left(\left(\sum_{j\in\mathcal{V}(i)} \lambda_t(i,j)\left(u_j^{T,r}(t) - u_i^{T,r}(t)\right)\right) - L\left(i, (\lambda_t(i,j))_{j\in\mathcal{V}(i)}\right)\right), \end{aligned}$$

Hamilton-Jacobi / Bellman equations

Hamilton-Jacobi / Bellman equations

Because

$$u_i^{T,r}(T) = g(i), \quad \forall i \in \mathcal{I},$$

Hamilton-Jacobi / Bellman equations

Because

$$u_i^{T,r}(T) = g(i), \quad \forall i \in \mathcal{I},$$

we are interested in the system of ODEs:

$$\forall i \in \mathcal{I}, \quad 0 = \frac{d}{dt} V_i^{T,r}(t) - r V_i^{T,r}(t) + \sup_{(\lambda_{ij})_{j \in \mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|}} \left(\left(\sum_{j \in \mathcal{V}(i)} \lambda_{ij} \left(V_j^{T,r}(t) - V_i^{T,r}(t) \right) \right) - L\left(i, (\lambda_{ij})_{j \in \mathcal{V}(i)}\right) \right)$$

with terminal condition $V_i^{T,r}(T) = g(i), \quad \forall i \in \mathcal{I}$.

Hamilton-Jacobi / Bellman equations

Hamilton-Jacobi / Bellman equations

To simplify notations, we introduce the Hamiltonian functions associated with the cost functions $(L(i,\cdot))_{i\in\mathcal{I}}$:

$$\forall i \in \mathcal{I}, H(i,\cdot) : p \in \mathbb{R}^{|\mathcal{V}(i)|} \mapsto H(i,p)$$

where

$$H(i,p) = \sup_{(\lambda_{ij})_{j\in\mathcal{V}(i)}\in\mathbb{R}_+^{|\mathcal{V}(i)|}} \left(\left(\sum_{j\in\mathcal{V}(i)} \lambda_{ij} p_j \right) - L\left(i, (\lambda_{ij})_{j\in\mathcal{V}(i)}\right) \right).$$

Hamilton-Jacobi / Bellman equations

Hamilton-Jacobi / Bellman equations

The ODEs then write:

$\forall (i,t) \in \mathcal{I} \times [0,T]$,

$$\frac{d}{dt} V_i^{T,r}(t) - r V_i^{T,r}(t) + H\left(i, \left(V_j^{T,r}(t) - V_i^{T,r}(t)\right)_{j \in \mathcal{V}(i)}\right) = 0$$

with terminal condition $V_i^{T,r}(T) = g(i), \quad \forall i \in \mathcal{I}$.

Hamilton-Jacobi / Bellman equations

The ODEs then write:

$\forall (i,t) \in \mathcal{I} \times [0,T]$,

$$\frac{d}{dt} V_i^{T,r}(t) - rV_i^{T,r}(t) + H\left(i, \left(V_j^{T,r}(t) - V_i^{T,r}(t)\right)_{j \in \mathcal{V}(i)}\right) = 0$$

with terminal condition $V_i^{T,r}(T) = g(i), \quad \forall i \in \mathcal{I}$.

Our goal now

Prove existence (and uniqueness) on $\mathcal{I} \times [0,T]$.

Hamilton-Jacobi / Bellman equations

The ODEs then write:

$\forall (i,t) \in \mathcal{I} \times [0,T]$,

$$\frac{d}{dt} V_i^{T,r}(t) - rV_i^{T,r}(t) + H\left(i, \left(V_j^{T,r}(t) - V_i^{T,r}(t)\right)_{j\in\mathcal{V}(i)}\right) = 0$$

with terminal condition $V_i^{T,r}(T) = g(i), \quad \forall i \in \mathcal{I}$.

Our goal now

Prove existence (and uniqueness) on $\mathcal{I} \times [0,T]$.

The solution will be the value function $(u_i^{T,r})_{i\in\mathcal{I}}$ and the optimal controls of an agent in state i at time t given by any maximizer of

$$\left(\sum_{j\in\mathcal{V}(i)} \lambda_{ij}\left(u_j^{T,r}(t) - u_i^{T,r}(t)\right)\right) - L\left(i, (\lambda_{ij})_{j\in\mathcal{V}(i)}\right)$$

How to prove existence / uniqueness for ODEs?

How to prove existence / uniqueness for ODEs?

Main theorems

- For local (in time) existence and uniqueness: Cauchy-Lipschitz / Picard-Lindelöf theorem $\rightarrow$ requires locally Lipschitz properties of H (with respect to p).

How to prove existence / uniqueness for ODEs?

Main theorems

- For local (in time) existence and uniqueness: Cauchy-Lipschitz / Picard-Lindelöf theorem $\rightarrow$ requires locally Lipschitz properties of H (with respect to p).
- For global (in time) existence and uniqueness: Global versions of Cauchy-Lipschitz / Picard-Lindelöf theorem $\rightarrow$ requires Lipschitz properties of H (with respect to p) – too much here.

How to prove existence / uniqueness for ODEs?

Main theorems

- For local (in time) existence and uniqueness: Cauchy-Lipschitz / Picard-Lindelöf theorem $\rightarrow$ requires locally Lipschitz properties of H (with respect to p).
- For global (in time) existence and uniqueness: Global versions of Cauchy-Lipschitz / Picard-Lindelöf theorem $\rightarrow$ requires Lipschitz properties of H (with respect to p) – too much here.
- For local (in time) existence only: Peano existence theorem $\rightarrow$ requires continuity of H (with respect to p) – we can do better here.

How to prove existence / uniqueness for ODEs?

Main theorems

- For local (in time) existence and uniqueness: Cauchy-Lipschitz / Picard-Lindelöf theorem $\rightarrow$ requires locally Lipschitz properties of H (with respect to p).
- For global (in time) existence and uniqueness: Global versions of Cauchy-Lipschitz / Picard-Lindelöf theorem $\rightarrow$ requires Lipschitz properties of H (with respect to p) – too much here.
- For local (in time) existence only: Peano existence theorem $\rightarrow$ requires continuity of H (with respect to p) – we can do better here.

From local to (half-)global existence

- Monotonicity properties
- Comparison principles
- A priori estimates
- etc.

Assumptions on the function L

Assumptions on the function L

1. Non-degeneracy:
   $\forall i \in \mathcal{I}, \exists \left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)} \in \mathbb{R}_+^{*|\mathcal{V}(i)|}, L\left(i, \left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)}\right) < +\infty.$

Assumptions on the function L

1. Non-degeneracy:
   $\forall i \in \mathcal{I}, \exists \left(\lambda_{ij}\right)_{j \in \mathcal{V}(i)} \in \mathbb{R}_+^{*|\mathcal{V}(i)|}, L\left(i, \left(\lambda_{ij}\right)_{j \in \mathcal{V}(i)}\right) < +\infty.$
2. Lower semi-continuity: $\forall i \in \mathcal{I}$, $L(i, \cdot)$ is lower semi-continuous.

Assumptions on the function L

1. Non-degeneracy:
   $\forall i \in \mathcal{I}, \exists \left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)} \in \mathbb{R}_+^{*|\mathcal{V}(i)|}, L\left(i, \left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)}\right) < +\infty.$
2. Lower semi-continuity: $\forall i \in \mathcal{I}$, $L(i,\cdot)$ is lower semi-continuous.
3. Asymptotic super-linearity:

$$\forall i \in \mathcal{I}, \lim_{\left\|\left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)}\right\|_\infty \to +\infty} \frac{L\left(i, \left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)}\right)}{\left\|\left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)}\right\|_\infty} = +\infty.$$

Assumptions on the function L

1. Non-degeneracy:
   $\forall i \in \mathcal{I}, \exists \left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)} \in \mathbb{R}_+^{*|\mathcal{V}(i)|}, L\left(i, \left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)}\right) < +\infty.$
2. Lower semi-continuity: $\forall i \in \mathcal{I}$, $L(i,\cdot)$ is lower semi-continuous.
3. Asymptotic super-linearity:

$$\forall i \in \mathcal{I}, \lim_{\left\|\left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)}\right\|_\infty \to +\infty} \frac{L\left(i, \left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)}\right)}{\left\|\left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)}\right\|_\infty} = +\infty.$$

4. Boundedness from below (not really an assumption):$\exists \underline{C} \in \mathbb{R}$,
   $\forall i \in \mathcal{I}$, $\forall \left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|}$, $L\left(i, \left(\lambda_{ij}\right)_{j\in\mathcal{V}(i)}\right) \geq \underline{C}$.

Consequences for the function H

Proposition

$\forall i \in \mathcal{I}$, the function $H(i,\cdot)$ is finite and verifies the following properties:

- $\forall p = (p_j)_{j\in\mathcal{V}(i)} \in \mathbb{R}^{|\mathcal{V}(i)|}, \exists \left(\lambda^*_{ij}\right)_{j\in\mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|}$,

$$H(i,p) = \left(\sum_{j\in\mathcal{V}(i)} \lambda^*_{ij} p_j\right) - L\left(i, \left(\lambda^*_{ij}\right)_{j\in\mathcal{V}(i)}\right).$$

- *$H(i,\cdot)$ is convex on $\mathbb{R}^{|\mathcal{V}(i)|}$. In particular it is locally Lipschitz.*
- *$H(i,\cdot)$ is non-decreasing with respect to each coordinate.*

Consequences for the function H

Proposition

$\forall i \in \mathcal{I}$, the function $H(i, \cdot)$ is finite and verifies the following properties:

- $\forall p = (p_j)_{j \in \mathcal{V}(i)} \in \mathbb{R}^{|\mathcal{V}(i)|}, \exists \left(\lambda^*_{ij}\right)_{j \in \mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|}$,

$$H(i, p) = \left(\sum_{j \in \mathcal{V}(i)} \lambda^*_{ij} p_j \right) - L\left(i, \left(\lambda^*_{ij}\right)_{j \in \mathcal{V}(i)}\right).$$

- *$H(i, \cdot)$ is convex on $\mathbb{R}^{|\mathcal{V}(i)|}$. In particular it is locally Lipschitz.*
- *$H(i, \cdot)$ is non-decreasing with respect to each coordinate.*

We can therefore use Picard-Lindelöf theorem to get (local) existence and uniqueness over an interval $(\tau, T]$

Consequences for the function H

Proposition

$\forall i \in \mathcal{I}$, the function $H(i,\cdot)$ is finite and verifies the following properties:

- $\forall p = (p_j)_{j\in\mathcal{V}(i)} \in \mathbb{R}^{|\mathcal{V}(i)|}, \exists \left(\lambda^*_{ij}\right)_{j\in\mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|}$,

$$H(i,p) = \left(\sum_{j\in\mathcal{V}(i)} \lambda^*_{ij} p_j\right) - L\left(i, \left(\lambda^*_{ij}\right)_{j\in\mathcal{V}(i)}\right).$$

- *$H(i,\cdot)$ is convex on $\mathbb{R}^{|\mathcal{V}(i)|}$. In particular it is locally Lipschitz.*
- *$H(i,\cdot)$ is non-decreasing with respect to each coordinate.*

We can therefore use Picard-Lindelöf theorem to get (local) existence and uniqueness over an interval $(\tau, T]$

$\rightarrow$ How to be sure that $[0, T]$ is included?

Sketch of proof

Proof.

Sketch of proof

Proof.

- Because of non-degeneracy $H(i, p) \neq -\infty$.

Sketch of proof

Proof.

- Because of non-degeneracy $H(i, p) \neq -\infty$.
- Because of asymptotic super-linearity, there is a compact set $\mathcal{C}$ such that

$$\sup_{(\lambda_{ij})_{j \in \mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|}} \left(\left(\sum_{j \in \mathcal{V}(i)} \lambda_{ij} p_j \right) - L\left(i, (\lambda_{ij})_{j \in \mathcal{V}(i)}\right) \right)$$

$$= \sup_{(\lambda_{ij})_{j \in \mathcal{V}(i)} \in \mathcal{C}} \left(\left(\sum_{j \in \mathcal{V}(i)} \lambda_{ij} p_j \right) - L\left(i, (\lambda_{ij})_{j \in \mathcal{V}(i)}\right) \right)$$

Sketch of proof

Proof.

- Because of non-degeneracy $H(i,p) \neq -\infty$.
- Because of asymptotic super-linearity, there is a compact set $\mathcal{C}$ such that

$$\sup_{(\lambda_{ij})_{j\in\mathcal{V}(i)}\in\mathbb{R}_+^{|\mathcal{V}(i)|}} \left(\left(\sum_{j\in\mathcal{V}(i)} \lambda_{ij} p_j\right) - L\left(i, (\lambda_{ij})_{j\in\mathcal{V}(i)}\right)\right)$$

$$= \sup_{(\lambda_{ij})_{j\in\mathcal{V}(i)}\in\mathcal{C}} \left(\left(\sum_{j\in\mathcal{V}(i)} \lambda_{ij} p_j\right) - L\left(i, (\lambda_{ij})_{j\in\mathcal{V}(i)}\right)\right)$$

- Because $L(i,\cdot)$ is l.s.c, the supremum is reached.

Sketch of proof

Proof.

- Because of non-degeneracy $H(i,p) \neq -\infty$.
- Because of asymptotic super-linearity, there is a compact set $\mathcal{C}$ such that

$$\sup_{(\lambda_{ij})_{j\in\mathcal{V}(i)}\in\mathbb{R}_+^{|\mathcal{V}(i)|}} \left(\left(\sum_{j\in\mathcal{V}(i)} \lambda_{ij} p_j\right) - L\left(i, (\lambda_{ij})_{j\in\mathcal{V}(i)}\right)\right)$$
$$= \sup_{(\lambda_{ij})_{j\in\mathcal{V}(i)}\in\mathcal{C}} \left(\left(\sum_{j\in\mathcal{V}(i)} \lambda_{ij} p_j\right) - L\left(i, (\lambda_{ij})_{j\in\mathcal{V}(i)}\right)\right)$$

- Because $L(i,\cdot)$ is l.s.c, the supremum is reached.
- Convexity of $H(i,\cdot)$ derives from the definition of $H(i,\cdot)$ as a supremum of affine functions.

Sketch of proof

Proof.

- Because of non-degeneracy $H(i,p) \neq -\infty$.
- Because of asymptotic super-linearity, there is a compact set $\mathcal{C}$ such that

$$\sup_{(\lambda_{ij})_{j\in\mathcal{V}(i)}\in\mathbb{R}_+^{|\mathcal{V}(i)|}} \left(\left(\sum_{j\in\mathcal{V}(i)} \lambda_{ij}p_j\right) - L\left(i,(\lambda_{ij})_{j\in\mathcal{V}(i)}\right)\right)$$
$$= \sup_{(\lambda_{ij})_{j\in\mathcal{V}(i)}\in\mathcal{C}} \left(\left(\sum_{j\in\mathcal{V}(i)} \lambda_{ij}p_j\right) - L\left(i,(\lambda_{ij})_{j\in\mathcal{V}(i)}\right)\right)$$

- Because $L(i,\cdot)$ is l.s.c, the supremum is reached.
- Convexity of $H(i,\cdot)$ derives from the definition of $H(i,\cdot)$ as a supremum of affine functions.
- Monotonicity of $H(i,\cdot)$ derives from the fact that the intensities $(\lambda_{ij})_{j\in\mathcal{V}(i)}$ are nonnegative.

□

From local to (half-)global existence

Proposition (Comparison principle)

Let $t' \in (-\infty, T)$. Let $(v_i)_{i\in\mathcal{I}}$ and $(w_i)_{i\in\mathcal{I}}$ be two continuously differentiable functions on $[t', T]$ such that

$$\frac{d}{dt}v_i(t) - rv_i(t) + H\left(i, (v_j(t) - v_i(t))_{j\in\mathcal{V}(i)}\right) \geq 0, \forall (i,t) \in \mathcal{I} \times [t', T],$$
$$\frac{d}{dt}w_i(t) - rw_i(t) + H\left(i, (w_j(t) - w_i(t))_{j\in\mathcal{V}(i)}\right) \leq 0, \forall (i,t) \in \mathcal{I} \times [t', T],$$

and $v_i(T) \leq w_i(T), \forall i \in \mathcal{I}$.

Then $v_i(t) \leq w_i(t)$, $\forall (i,t) \in \mathcal{I} \times [t', T]$.

Proof of the comparison principle

Proof.

Proof of the comparison principle

Proof.

Let $\varepsilon > 0$.

Proof of the comparison principle

Proof.

Let $\varepsilon > 0$.

Let us define

$$z : (i,t) \in \mathcal{I} \times [t', T] \mapsto z_i(t) = e^{-rt}(v_i(t) - w_i(t) - \varepsilon(T - t)).$$

Proof of the comparison principle

Proof.

Let $\varepsilon > 0$.

Let us define

$$z : (i,t) \in \mathcal{I} \times [t', T] \mapsto z_i(t) = e^{-rt}(v_i(t) - w_i(t) - \varepsilon(T-t)).$$

We have

$$\begin{aligned}
\frac{d}{dt} z_i(t) &= -re^{-rt}(v_i(t) - w_i(t) - \varepsilon(T-t)) + e^{-rt}\left(\frac{d}{dt}v_i(t) - \frac{d}{dt}w_i(t) + \varepsilon\right) \\
&= e^{-rt}\left(\left(\frac{d}{dt}v_i(t) - rv_i(t)\right) - \left(\frac{d}{dt}w_i(t) - rw_i(t)\right) + \varepsilon + r\varepsilon(T-t)\right) \\
&\geq e^{-rt}\left(-H\left(i, (v_j(t) - v_i(t))_{j\in\mathcal{V}(i)}\right) + H\left(i, (w_j(t) - w_i(t))_{j\in\mathcal{V}(i)}\right)\right) \\
&+ e^{-rt}\left(\varepsilon + r\varepsilon(T-t)\right).
\end{aligned}$$

Proof of the comparison principle

Proof.

Proof of the comparison principle

Proof.

Let us choose $(i^*, t^*) \in \mathcal{I} \times [t', T]$ maximizing z.

Proof of the comparison principle

Proof.

Let us choose $(i^*, t^*) \in \mathcal{I} \times [t', T]$ maximizing z.

We now show by contradiction that $t^* = T$.

Proof of the comparison principle

Proof.

Let us choose $(i^*, t^*) \in \mathcal{I} \times [t', T]$ maximizing z.
We now show by contradiction that $t^* = T$.

$$t^* < T \implies \frac{d}{dt} z_{i^*}(t^*) \leq 0 \implies$$

$$\begin{aligned} H\left(i^*, \left(\left(v_j(t^*) - v_{i^*}(t^*)\right)\right)_{j \in \mathcal{V}(i^*)}\right) \quad \geq \quad & H\left(i^*, \left(\left(w_j(t^*) - w_{i^*}(t^*)\right)\right)_{j \in \mathcal{V}(i^*)}\right) \\ & + \varepsilon + r\varepsilon(T - t^*). \end{aligned}$$

Proof of the comparison principle

Proof.

Let us choose $(i^*, t^*) \in \mathcal{I} \times [t', T]$ maximizing z.
We now show by contradiction that $t^* = T$.

$$t^* < T \implies \frac{d}{dt} z_{i^*}(t^*) \leq 0 \implies$$

$$\begin{aligned} H\left(i^*, ((v_j(t^*) - v_{i^*}(t^*))_{j \in \mathcal{V}(i^*)}\right) \quad &\geq \quad H\left(i^*, ((w_j(t^*) - w_{i^*}(t^*))_{j \in \mathcal{V}(i^*)}\right) \\ &\quad +\varepsilon + r\varepsilon(T - t^*). \end{aligned}$$

By definition of (i^*, t^*), we know that

$$\forall j \in \mathcal{V}(i^*), v_j(t^*) - w_j(t^*) \leq v_{i^*}(t^*) - w_{i^*}(t^*)$$

i.e.

$$\forall j \in \mathcal{V}(i^*), v_j(t^*) - v_{i^*}(t^*) \leq w_j(t^*) - w_{i^*}(t^*).$$

Proof of the comparison principle

Proof.

Proof of the comparison principle

Proof.

From the monotonicity of $H(i^*, \cdot)$, it follows that

$$H\left(i^*, \left(v_j\left(t^*\right) - v_{i^*}\left(t^*\right)\right)_{j\in\mathcal{V}(i^*)}\right) \leq H\left(i^*, \left(w_j\left(t^*\right) - w_{i^*}\left(t^*\right)\right)_{j\in\mathcal{V}(i^*)}\right).$$

Proof of the comparison principle

Proof.

From the monotonicity of $H(i^*, \cdot)$, it follows that

$$H\left(i^*, (v_j(t^*) - v_{i^*}(t^*))_{j \in \mathcal{V}(i^*)}\right) \leq H\left(i^*, (w_j(t^*) - w_{i^*}(t^*))_{j \in \mathcal{V}(i^*)}\right).$$

This contradicts the above inequality.

Proof of the comparison principle

Proof.

From the monotonicity of $H(i^*, \cdot)$, it follows that

$$H\left(i^*, \left(v_j\left(t^*\right) - v_{i^*}\left(t^*\right)\right)_{j \in \mathcal{V}(i^*)}\right) \leq H\left(i^*, \left(w_j\left(t^*\right) - w_{i^*}\left(t^*\right)\right)_{j \in \mathcal{V}(i^*)}\right).$$

This contradicts the above inequality.
Therefore, $t^* = T$,

Proof of the comparison principle

Proof.

From the monotonicity of $H(i^*, \cdot)$, it follows that

$$H\left(i^*, (v_j(t^*) - v_{i^*}(t^*))_{j \in \mathcal{V}(i^*)}\right) \leq H\left(i^*, (w_j(t^*) - w_{i^*}(t^*))_{j \in \mathcal{V}(i^*)}\right).$$

This contradicts the above inequality.
Therefore, $t^* = T$, and we have:

$$\forall (i,t) \in \mathcal{I} \times [t', T], \quad z_i(t) \leq z_{i^*}(T) = e^{-rT}(v_{i^*}(T) - w_{i^*}(T)) \leq 0.$$

Proof of the comparison principle

Proof.

From the monotonicity of $H(i^*, \cdot)$, it follows that

$$H\left(i^*, \left(v_j\left(t^*\right) - v_{i^*}\left(t^*\right)\right)_{j\in\mathcal{V}(i^*)}\right) \leq H\left(i^*, \left(w_j\left(t^*\right) - w_{i^*}\left(t^*\right)\right)_{j\in\mathcal{V}(i^*)}\right).$$

This contradicts the above inequality.
Therefore, $t^* = T$, and we have:

$$\forall (i,t) \in \mathcal{I} \times [t', T], \quad z_i(t) \leq z_{i^*}(T) = e^{-rT}(v_{i^*}(T) - w_{i^*}(T)) \leq 0.$$

Therefore, $\forall (i,t) \in \mathcal{I} \times [t', T], \quad v_i(t) \leq w_i(t) + \varepsilon(T - t)$

Proof of the comparison principle

Proof.

From the monotonicity of $H(i^*, \cdot)$, it follows that

$$H\left(i^*, \left(v_j\left(t^*\right) - v_{i^*}\left(t^*\right)\right)_{j \in \mathcal{V}(i^*)}\right) \leq H\left(i^*, \left(w_j\left(t^*\right) - w_{i^*}\left(t^*\right)\right)_{j \in \mathcal{V}(i^*)}\right).$$

This contradicts the above inequality.
Therefore, $t^* = T$, and we have:

$$\forall (i,t) \in \mathcal{I} \times [t', T], \quad z_i(t) \leq z_{i^*}(T) = e^{-rT}(v_{i^*}(T) - w_{i^*}(T)) \leq 0.$$

Therefore, $\forall (i,t) \in \mathcal{I} \times [t', T], \quad v_i(t) \leq w_i(t) + \varepsilon(T - t)$ and we conclude by sending ε to 0. □

Existence and uniqueness theorem

Theorem ((Half-)Global existence and uniqueness)

There exists a unique solution $\left(V_i^{T,r}\right)_{i\in\mathcal{I}}$ *on* $(-\infty, T]$ *to the Hamilton-Jacobi/Bellman equation*

$$\forall i \in \mathcal{I}, \quad 0 = \frac{d}{dt}V_i^{T,r}(t) - rV_i^{T,r}(t) + \sup_{(\lambda_{ij})_{j\in\mathcal{V}(i)}\in\mathbb{R}_+^{|\mathcal{V}(i)|}} \left(\left(\sum_{j\in\mathcal{V}(i)} \lambda_{ij}\left(V_j^{T,r}(t) - V_i^{T,r}(t)\right)\right) - L\left(i, (\lambda_{ij})_{j\in\mathcal{V}(i)}\right)\right)$$

with terminal condition $V_i^{T,r}(T) = g(i), \quad \forall i \in \mathcal{I}.$

Proof of the existence and uniqueness theorem

Proof.

Proof of the existence and uniqueness theorem

Proof.

$\forall i \in \mathcal{I}$, the function $H(i, \cdot)$ is locally Lipschitz. Therefore by Picard-Lindelöf theorem there exists a (left-)maximal solution $\left(V_i^{T,r}\right)_{i \in \mathcal{I}}$ defined over $(\tau^*, T]$, where $\tau^* \in [-\infty, T)$.

Proof of the existence and uniqueness theorem

Proof.

$\forall i \in \mathcal{I}$, the function $H(i, \cdot)$ is locally Lipschitz. Therefore by Picard-Lindelöf theorem there exists a (left-)maximal solution $\left(V_i^{T,r}\right)_{i\in\mathcal{I}}$ defined over $(\tau^*, T]$, where $\tau^* \in [-\infty, T)$.

Our goal is to prove by contradiction that $\tau^* = -\infty$.

Proof of the existence and uniqueness theorem

Proof.

$\forall i \in \mathcal{I}$, the function $H(i, \cdot)$ is locally Lipschitz. Therefore by Picard-Lindelöf theorem there exists a (left-)maximal solution $\left(V_i^{T,r}\right)_{i\in\mathcal{I}}$ defined over $(\tau^*, T]$, where $\tau^* \in [-\infty, T)$.

Our goal is to prove by contradiction that $\tau^* = -\infty$.

For $C \in \mathbb{R}$, let us consider

$$v^C : (i,t) \in \mathcal{I} \times (\tau^*, T] \mapsto v_i^C(t) = e^{-r(T-t)}\left(g(i) + C(T-t)\right).$$

Proof of the existence and uniqueness theorem

Proof.

$\forall i \in \mathcal{I}$, the function $H(i, \cdot)$ is locally Lipschitz. Therefore by Picard-Lindelöf theorem there exists a (left-)maximal solution $\left(V_i^{T,r}\right)_{i\in\mathcal{I}}$ defined over $(\tau^*, T]$, where $\tau^* \in [-\infty, T)$.

Our goal is to prove by contradiction that $\tau^* = -\infty$.

For $C \in \mathbb{R}$, let us consider

$$v^C : (i,t) \in \mathcal{I} \times (\tau^*, T] \mapsto v_i^C(t) = e^{-r(T-t)} \left(g(i) + C(T-t)\right).$$

We see that

$$\begin{aligned} &\frac{d}{dt} v_i^C(t) - r v_i^C(t) + H\left(i, \left(v_j^C(t) - v_i^C(t)\right)_{j\in\mathcal{V}(i)}\right) \\ = &- Ce^{-r(T-t)} + H\left(i, e^{-r(T-t)} \left(g(j) - g(i)\right)_{j\in\mathcal{V}(i)}\right) \end{aligned}$$

Proof of the existence and uniqueness theorem

Proof.

Proof of the existence and uniqueness theorem

Proof.

If τ^* is finite, the function

$$(i,t) \in \mathcal{I} \times (\tau^*, T] \mapsto e^{r(T-t)} H\left(i, e^{-r(T-t)}(g(j)-g(i))_{j\in\mathcal{V}(i)}\right)$$

is bounded.

Proof of the existence and uniqueness theorem

Proof.

If τ^* is finite, the function

$$(i,t) \in \mathcal{I} \times (\tau^*, T] \mapsto e^{r(T-t)} H\left(i, e^{-r(T-t)}(g(j)-g(i))_{j \in \mathcal{V}(i)}\right)$$

is bounded.

So, there exist C_1 and C_2 such that $\forall (i,t) \in \mathcal{I} \times (\tau^*, T]$,

$$\begin{aligned} & -C_1 e^{-r(T-t)} + H\left(i, e^{-r(T-t)}(g(j)-g(i))_{j \in \mathcal{V}(i)}\right) \geq 0, \quad \text{and} \\ & -C_2 e^{-r(T-t)} + H\left(i, e^{-r(T-t)}(g(j)-g(i))_{j \in \mathcal{V}(i)}\right) \leq 0. \end{aligned}$$

Proof of the existence and uniqueness theorem

Proof.

If τ^* is finite, the function

$$(i,t) \in \mathcal{I} \times (\tau^*, T] \mapsto e^{r(T-t)} H\left(i, e^{-r(T-t)}(g(j)-g(i))_{j\in\mathcal{V}(i)}\right)$$

is bounded.

So, there exist C_1 and C_2 such that $\forall (i,t) \in \mathcal{I} \times (\tau^*, T]$,

$$\begin{aligned} &-C_1 e^{-r(T-t)} + H\left(i, e^{-r(T-t)}(g(j)-g(i))_{j\in\mathcal{V}(i)}\right) \geq 0, \quad \text{and} \\ &-C_2 e^{-r(T-t)} + H\left(i, e^{-r(T-t)}(g(j)-g(i))_{j\in\mathcal{V}(i)}\right) \leq 0. \end{aligned}$$

Applying the above comparison principle over any interval $[t', T] \subset (\tau^*, T]$, we obtain:

$$\forall (i,t) \in \mathcal{I} \times [t', T], \quad v_i^{C_1}(t) \leq V_i^{T,r}(t) \leq v_i^{C_2}(t).$$

Proof of the existence and uniqueness theorem

Proof.

Proof of the existence and uniqueness theorem

Proof.

By sending t' to τ^* we obtain

$$\forall (i,t) \in \mathcal{I} \times (\tau^*, T], \quad v_i^{C_1}(t) \leq V_i^{T,r}(t) \leq v_i^{C_2}(t).$$

Proof of the existence and uniqueness theorem

Proof.

By sending t' to τ^* we obtain

$$\forall (i,t) \in \mathcal{I} \times (\tau^*, T], \quad v_i^{C_1}(t) \leq V_i^{T,r}(t) \leq v_i^{C_2}(t).$$

In particular, τ^* finite implies that the functions $\left(V_i^{T,r}\right)_{i\in\mathcal{I}}$ are bounded... in contradiction with the maximality of τ^*.

□

Proof of the existence and uniqueness theorem

Proof.

By sending t' to τ^* we obtain

$$\forall (i,t) \in \mathcal{I} \times (\tau^*, T], \quad v_i^{C_1}(t) \leq V_i^{T,r}(t) \leq v_i^{C_2}(t).$$

In particular, τ^* finite implies that the functions $\left(V_i^{T,r}\right)_{i\in\mathcal{I}}$ are bounded... in contradiction with the maximality of τ^*.

□

In the proof of the above results, the convexity of the Hamiltonian functions $(H(i,\cdot))_{i\in\mathcal{I}}$ does not play any role.

Proof of the existence and uniqueness theorem

Proof.

By sending t' to τ^* we obtain

$$\forall (i,t) \in \mathcal{I} \times (\tau^*, T], \quad v_i^{C_1}(t) \leq V_i^{T,r}(t) \leq v_i^{C_2}(t).$$

In particular, τ^* finite implies that the functions $\left(V_i^{T,r}\right)_{i\in\mathcal{I}}$ are bounded... in contradiction with the maximality of τ^*.

□

In the proof of the above results, the convexity of the Hamiltonian functions $(H(i,\cdot))_{i\in\mathcal{I}}$ does not play any role.

The results indeed hold as soon as the Hamiltonian functions are locally Lipschitz and non-decreasing with respect to each coordinate.

Going back to the optimal control problem

Going back to the optimal control problem

Theorem (Verification theorem)

Going back to the optimal control problem

Theorem (Verification theorem)

- $\forall (i,t) \in \mathcal{I} \times [0,T], u_i^{T,r}(t) = V_i^{T,r}(t)$.

Going back to the optimal control problem

Theorem (Verification theorem)

- $\forall (i,t) \in \mathcal{I} \times [0,T], u_i^{T,r}(t) = V_i^{T,r}(t)$.
- *The optimal controls are given by any feedback control function verifying for all $i \in \mathcal{I}$, for all $j \in \mathcal{V}(i)$, and for all $t \in [0,T]$,*

$$\lambda_t^*(i,j) \in \underset{(\lambda_{ij})_{j\in\mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|}}{\operatorname{argmax}} \left(\left(\sum_{j\in\mathcal{V}(i)} \lambda_{ij} \left(u_j^{T,r}(t) - u_i^{T,r}(t) \right) \right) - L\left(i, (\lambda_{ij})_{j\in\mathcal{V}(i)}\right) \right).$$

Going back to the optimal control problem

Theorem (Verification theorem)

- $\forall (i,t) \in \mathcal{I} \times [0,T], u_i^{T,r}(t) = V_i^{T,r}(t).$
- *The optimal controls are given by any feedback control function verifying for all $i \in \mathcal{I}$, for all $j \in \mathcal{V}(i)$, and for all $t \in [0,T]$,*

$$\lambda_t^*(i,j) \in \underset{(\lambda_{ij})_{j\in\mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|}}{\operatorname{argmax}} \left(\left(\sum_{j\in\mathcal{V}(i)} \lambda_{ij} \left(u_j^{T,r}(t) - u_i^{T,r}(t) \right) \right) - L\left(i, (\lambda_{ij})_{j\in\mathcal{V}(i)}\right) \right).$$

The above argmax is a always singleton if the Hamiltonian functions $(H(i,\cdot))_i$ are differentiable (which is guaranteed if $(L(i,\cdot))_i$ are convex functions that are strictly convex on their domain).

What's next?

What's next?

- In many problems, there is no final time T

What's next?

- In many problems, there is no final time T (e.g. no natural T in the (re)commerce problem)

What's next?

- In many problems, there is no final time T (e.g. no natural T in the (re)commerce problem)
- What happens when $T \to \infty$?

What's next?

- In many problems, there is no final time T (e.g. no natural T in the (re)commerce problem)
- What happens when $T \to \infty$?
    - What is the asymptotic behavior of the value function?

What's next?

- In many problems, there is no final time T (e.g. no natural T in the (re)commerce problem)
- What happens when $T \to \infty$?
  - What is the asymptotic behavior of the value function?
  - What is the asymptotic behavior of the optimal controls / optimal transition intensities?

What's next?

- In many problems, there is no final time T (e.g. no natural T in the (re)commerce problem)
- What happens when $T \to \infty$?
  - What is the asymptotic behavior of the value function?
  - What is the asymptotic behavior of the optimal controls / optimal transition intensities?

Two cases: $r > 0$ and $r = 0$

A general theory for optimal control on graphs – Asymptotics when $r > 0$

Study of the $r > 0$ case

Study of the $r > 0$ case

Proposition

$$\exists (u_i^r)_{i\in\mathcal{I}} \in \mathbb{R}^N, \forall (i,t) \in \mathcal{I} \times \mathbb{R}_+, \lim_{T\to+\infty} u_i^{T,r}(t) = u_i^r.$$

Furthermore, $(u_i^r)_{i\in\mathcal{I}}$ *satisfies the following stationary Bellman equation:*

$$-ru_i^r + H\left(i, \left(u_j^r - u_i^r\right)_{j\in\mathcal{V}(i)}\right) = 0, \quad \forall i \in \mathcal{I}.$$

Study of the $r > 0$ case

Study of the $r > 0$ case

Proof.

Let us define

$$u_i^r = \sup_\lambda \mathbb{E}\left[-\int_0^{+\infty} e^{-rt} L\left(X_t^{0,i,\lambda}, \left(\lambda_t\left(X_t^{0,i,\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda}\right)}\right) dt\right].$$

Study of the $r > 0$ case

Proof.

Let us define

$$u_i^r = \sup_{\lambda} \mathbb{E}\left[-\int_0^{+\infty} e^{-rt} L\left(X_t^{0,i,\lambda}, \left(\lambda_t\left(X_t^{0,i,\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda}\right)}\right) dt\right].$$

It is finite because L is bounded from below and because of the non-degeneracy assumption (we will see it more precisely later).

Study of the $r > 0$ case

Proof.

Let us define

$$u_i^r = \sup_{\lambda} \mathbb{E}\left[-\int_0^{+\infty} e^{-rt} L\left(X_t^{0,i,\lambda}, \left(\lambda_t \left(X_t^{0,i,\lambda}, j \right) \right)_{j \in \mathcal{V}\left(X_t^{0,i,\lambda}\right)} \right) dt \right].$$

It is finite because L is bounded from below and because of the non-degeneracy assumption (we will see it more precisely later).

Let us consider an optimal control λ^* of the optimal control problem over $[0, T]$.

Study of the $r > 0$ case

Proof.

Let us define

$$u_i^r = \sup_{\lambda} \mathbb{E}\left[-\int_0^{+\infty} e^{-rt} L\left(X_t^{0,i,\lambda}, \left(\lambda_t\left(X_t^{0,i,\lambda}, j\right)\right)_{j \in \mathcal{V}\left(X_t^{0,i,\lambda}\right)}\right) dt\right].$$

It is finite because L is bounded from below and because of the non-degeneracy assumption (we will see it more precisely later).

Let us consider an optimal control λ^* of the optimal control problem over $[0, T]$.

Let us define a control λ on $[0, +\infty)$ by:

Study of the $r > 0$ case

Proof.

Let us define

$$u_i^r = \sup_\lambda \mathbb{E}\left[-\int_0^{+\infty} e^{-rt} L\left(X_t^{0,i,\lambda}, \left(\lambda_t\left(X_t^{0,i,\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda}\right)}\right) dt\right].$$

It is finite because L is bounded from below and because of the non-degeneracy assumption (we will see it more precisely later).

Let us consider an optimal control λ^* of the optimal control problem over $[0, T]$.

Let us define a control λ on $[0, +\infty)$ by:

- $\lambda_t = \lambda_t^*$ for $t \in [0, T]$,

Study of the $r > 0$ case

Proof.

Let us define

$$u_i^r = \sup_{\lambda} \mathbb{E}\left[-\int_0^{+\infty} e^{-rt} L\left(X_t^{0,i,\lambda}, \left(\lambda_t\left(X_t^{0,i,\lambda}, j \right) \right)_{j \in \mathcal{V}(X_t^{0,i,\lambda})} \right) dt \right].$$

It is finite because L is bounded from below and because of the non-degeneracy assumption (we will see it more precisely later).

Let us consider an optimal control λ^* of the optimal control problem over $[0, T]$.

Let us define a control λ on $[0, +\infty)$ by:

- $\lambda_t = \lambda_t^*$ for $t \in [0, T]$,
- $\lambda_t(i,j) = \tilde{\lambda}(i,j)$ for $t > T$, where $\tilde{\lambda}$ is such that $L\left(i, (\tilde{\lambda}(i,j))_{j \in \mathcal{V}(i)} \right) < +\infty$.

Study of the $r > 0$ case

Study of the $r > 0$ case

Proof.

$$\begin{aligned}
u_i^r &\geq \mathbb{E}\left[-\int_0^\infty e^{-rt} L\left(X_t^{0,i,\lambda}, \left(\lambda_t\left(X_t^{0,i,\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda}\right)}\right) dt\right] \\
&\geq \mathbb{E}\left[-\int_0^T e^{-rt} L\left(X_t^{0,i,\lambda^*}, \left(\lambda_t^*\left(X_t^{0,i,\lambda^*}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda^*}\right)}\right) dt\right] \\
&+\mathbb{E}\left[-\int_T^\infty e^{-rt} L\left(X_t^{T,X_T^{0,i,\lambda^*},\lambda}, \left(\lambda_t\left(X_t^{T,X_T^{0,i,\lambda^*},\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{T,X_T^{0,i,\lambda^*},\lambda}\right)}\right) dt\right] \\
&\geq u_i^{T,r}(0) - e^{-rT} g\left(X_T^{0,i,\lambda^*}\right) \\
&+e^{-rT}\mathbb{E}\left[-\int_T^\infty e^{-r(t-T)} L\left(X_t^{T,X_T^{0,i,\lambda^*},\tilde\lambda}, \left(\tilde\lambda_t\left(X_t^{T,X_T^{0,i,\lambda^*},\tilde\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{T,X_T^{0,i,\lambda^*},\tilde\lambda}\right)}\right) dt\right] \\
&\geq u_i^{T,r}(0) - e^{-rT} g\left(X_T^{0,i,\lambda^*}\right) - \frac{M}{r} e^{-rT}.
\end{aligned}$$

Study of the $r > 0$ case

Proof.

$$
\begin{aligned}
u_i^r &\geq \mathbb{E}\left[-\int_0^\infty e^{-rt} L\left(X_t^{0,i,\lambda}, \left(\lambda_t\left(X_t^{0,i,\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda}\right)}\right) dt\right] \\
&\geq \mathbb{E}\left[-\int_0^T e^{-rt} L\left(X_t^{0,i,\lambda^*}, \left(\lambda_t^*\left(X_t^{0,i,\lambda^*}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda^*}\right)}\right) dt\right] \\
+\mathbb{E}&\left[-\int_T^\infty e^{-rt} L\left(X_t^{T,X_T^{0,i,\lambda^*},\lambda}, \left(\lambda_t\left(X_t^{T,X_T^{0,i,\lambda^*},\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{T,X_T^{0,i,\lambda^*},\lambda}\right)}\right) dt\right] \\
&\geq u_i^{T,r}(0) - e^{-rT} g\left(X_T^{0,i,\lambda^*}\right) \\
&+e^{-rT}\mathbb{E}\left[-\int_T^\infty e^{-r(t-T)} L\left(X_t^{T,X_T^{0,i,\lambda^*},\tilde{\lambda}}, \left(\tilde{\lambda}_t\left(X_t^{T,X_T^{0,i,\lambda^*},\tilde{\lambda}}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{T,X_T^{0,i,\lambda^*},\tilde{\lambda}}\right)}\right) dt\right] \\
&\geq u_i^{T,r}(0) - e^{-rT} g\left(X_T^{0,i,\lambda^*}\right) - \frac{M}{r} e^{-rT}.
\end{aligned}
$$

So $\limsup_{T\to+\infty} u_i^{T,r}(0) \leq u_i^r$.

Study of the $r > 0$ case

Study of the $r > 0$ case

Proof.

Let us consider $\varepsilon > 0$ and λ^ε such that

$$u_i^r - \varepsilon \leq \mathbb{E}\left[-\int_0^\infty e^{-rt} L\left(X_t^{0,i,\lambda^\varepsilon}, \left(\lambda_t^\varepsilon\left(X_t^{0,i,\lambda^\varepsilon}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda^\varepsilon}\right)}\right) dt\right].$$

Study of the $r > 0$ case

Proof.

Let us consider $\varepsilon > 0$ and λ^ε such that

$$u_i^r - \varepsilon \le \mathbb{E}\left[-\int_0^\infty e^{-rt} L\left(X_t^{0,i,\lambda^\varepsilon}, \left(\lambda_t^\varepsilon\left(X_t^{0,i,\lambda^\varepsilon}, j\right)\right)_{j\in\mathcal{V}(X_t^{0,i,\lambda^\varepsilon})}\right) dt\right].$$

We have

$$\begin{aligned} u_i^r - \varepsilon \quad &\le \quad \mathbb{E}\left[-\int_0^T e^{-rt} L\left(X_t^{0,i,\lambda^\varepsilon}, \left(\lambda_t^\varepsilon\left(X_t^{0,i,\lambda^\varepsilon}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda^\varepsilon}\right)}\right) dt\right] \\ &+\mathbb{E}\left[-\int_T^\infty e^{-rt} L\left(X_t^{T,X_T^{0,i,\lambda^\varepsilon},\lambda^\varepsilon}, \left(\lambda_t^\varepsilon\left(X_t^{T,X_T^{0,i,\lambda^\varepsilon},\lambda^\varepsilon}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{T,X_T^{0,i,\lambda^\varepsilon},\lambda^\varepsilon}\right)}\right) dt\right] \\ &\le \quad u_i^{T,r}(0) - e^{-rT} g\left(X_T^{0,i,\lambda^\varepsilon}\right) - e^{-rT}\frac{C}{r} \end{aligned}$$

Study of the $r > 0$ case

Proof.

Let us consider $\varepsilon > 0$ and λ^ε such that

$$u_i^r - \varepsilon \leq \mathbb{E}\left[-\int_0^\infty e^{-rt} L\left(X_t^{0,i,\lambda^\varepsilon}, \left(\lambda_t^\varepsilon\left(X_t^{0,i,\lambda^\varepsilon}, j\right)\right)_{j\in\mathcal{V}(X_t^{0,i,\lambda^\varepsilon})}\right) dt\right].$$

We have

$$\begin{aligned} u_i^r - \varepsilon \quad &\leq \quad \mathbb{E}\left[-\int_0^T e^{-rt} L\left(X_t^{0,i,\lambda^\varepsilon}, \left(\lambda_t^\varepsilon\left(X_t^{0,i,\lambda^\varepsilon}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda^\varepsilon}\right)}\right) dt\right] \\ &+\mathbb{E}\left[-\int_T^\infty e^{-rt} L\left(X_t^{T,X_T^{0,i,\lambda^\varepsilon},\lambda^\varepsilon}, \left(\lambda_t^\varepsilon\left(X_t^{T,X_T^{0,i,\lambda^\varepsilon},\lambda^\varepsilon}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{T,X_T^{0,i,\lambda^\varepsilon},\lambda^\varepsilon}\right)}\right) dt\right] \\ &\leq \quad u_i^{T,r}(0) - e^{-rT} g\left(X_T^{0,i,\lambda^\varepsilon}\right) - e^{-rT}\frac{C}{r} \end{aligned}$$

So $\liminf_{T\to+\infty} u_i^{T,r}(0) \geq u_i^r - \varepsilon$.

Study of the $r > 0$ case

Study of the $r > 0$ case

Proof.

By sending ε to 0, we obtain $\lim_{T \to +\infty} u_i^{T,r}(0) = u_i^r$.

Study of the $r > 0$ case

Proof.

By sending ε to 0, we obtain $\lim_{T\to+\infty} u_i^{T,r}(0) = u_i^r$.
We easily see that

$$\forall i \in \mathcal{I}, \forall s, t \in \mathbb{R}_+, \forall T > t, u_i^{T+s,r}(t) = u_i^{T+s-t,r}(0) = V_i^{T,r}(t-s).$$

Study of the $r > 0$ case

Proof.

By sending ε to 0, we obtain $\lim_{T\to+\infty} u_i^{T,r}(0) = u_i^r$.
We easily see that

$$\forall i \in \mathcal{I}, \forall s, t \in \mathbb{R}_+, \forall T > t, u_i^{T+s,r}(t) = u_i^{T+s-t,r}(0) = V_i^{T,r}(t-s).$$

Therefore

$$\forall (i,t) \in \mathcal{I} \times \mathbb{R}_+, \lim_{T\to+\infty} u_i^{T,r}(t) = u_i^r = \lim_{s\to-\infty} V_i^{T,r}(s)$$

Study of the $r > 0$ case

Proof.

By sending ε to 0, we obtain $\lim_{T\to+\infty} u_i^{T,r}(0) = u_i^r$.
We easily see that

$$\forall i \in \mathcal{I}, \forall s, t \in \mathbb{R}_+, \forall T > t, u_i^{T+s,r}(t) = u_i^{T+s-t,r}(0) = V_i^{T,r}(t-s).$$

Therefore

$$\forall (i,t) \in \mathcal{I} \times \mathbb{R}_+, \lim_{T\to+\infty} u_i^{T,r}(t) = u_i^r = \lim_{s\to-\infty} V_i^{T,r}(s)$$

Using the ODEs, we see that $\frac{d}{dt}\left(V_i^{T,r}\right)_{i\in\mathcal{I}}$ has a finite limit in $-\infty$.
But, then, that limit is equal to 0.

Study of the $r > 0$ case

Proof.

By sending ε to 0, we obtain $\lim_{T\to+\infty} u_i^{T,r}(0) = u_i^r$.
We easily see that

$$\forall i \in \mathcal{I}, \forall s, t \in \mathbb{R}_+, \forall T > t, u_i^{T+s,r}(t) = u_i^{T+s-t,r}(0) = V_i^{T,r}(t-s).$$

Therefore

$$\forall (i,t) \in \mathcal{I} \times \mathbb{R}_+, \lim_{T\to+\infty} u_i^{T,r}(t) = u_i^r = \lim_{s\to-\infty} V_i^{T,r}(s)$$

Using the ODEs, we see that $\frac{d}{dt}\left(V_i^{T,r}\right)_{i\in\mathcal{I}}$ has a finite limit in $-\infty$.
But, then, that limit is equal to 0.
By passing to the limit in the ODEs, we obtain

$$-ru_i^r + H\left(i, \left(u_j^r - u_i^r\right)_{j\in\mathcal{V}(i)}\right) = 0, \quad \forall i \in \mathcal{I}.$$

$\square$

The limit case $r \to 0$

What happens when $r \to 0$

What happens when $r \to 0$

For studying the asymptotic behavior (as $T \to +\infty$) in the case $r = 0$, a first step consists in studying what happens when $r \to 0$ in the above.

Our goal is to prove the following proposition:

Proposition

We have:

- $\exists \gamma \in \mathbb{R}, \forall i \in \mathcal{I}, \lim_{r \to 0} r u_i^r = \gamma$.
- *There exists a sequence* $(r_n)_{n\in\mathbb{N}}$ *converging towards* 0 *such that* $\forall i \in \mathcal{I}, \left(u_i^{r_n} - u_1^{r_n}\right)_{n\in\mathbb{N}}$ *is convergent.*
- *For all* $i \in \mathcal{I}$, *if* $\xi_i = \lim_{n\to+\infty} u_i^{r_n} - u_1^{r_n}$, *then we have*

$$-\gamma + H\left(i, (\xi_j - \xi_i)_{j\in\mathcal{V}(i)}\right) = 0.$$

A first lemma to study $r \to 0$

A first lemma to study $r \to 0$

Lemma

A first lemma to study $r \to 0$

Lemma

We have:

1. $\forall i \in \mathcal{I}$, $r \in \mathbb{R}_+^* \mapsto r u_i^r$ *is bounded;*
2. $\forall i \in \mathcal{I}$, $\forall j \in \mathcal{V}(i)$, $r \in \mathbb{R}_+^* \mapsto u_j^r - u_i^r$ *is bounded.*

A first lemma to study $r \to 0$

Lemma

We have:

1. $\forall i \in \mathcal{I}$, $r \in \mathbb{R}_+^* \mapsto r u_i^r$ *is bounded;*
2. $\forall i \in \mathcal{I}$, $\forall j \in \mathcal{V}(i)$, $r \in \mathbb{R}_+^* \mapsto u_j^r - u_i^r$ *is bounded.*

Proof.

Let us choose $(\lambda(i,j))_{i \in \mathcal{I}, j \in \mathcal{V}(i)} \in \mathcal{A}$ as in the non-degeneracy assumption.

A first lemma to study $r \to 0$

Lemma

We have:

1. $\forall i \in \mathcal{I}$, $r \in \mathbb{R}_+^* \mapsto r u_i^r$ *is bounded;*
2. $\forall i \in \mathcal{I}, \forall j \in \mathcal{V}(i)$, $r \in \mathbb{R}_+^* \mapsto u_j^r - u_i^r$ *is bounded.*

Proof.

Let us choose $(\lambda(i,j))_{i\in\mathcal{I}, j\in\mathcal{V}(i)} \in \mathcal{A}$ as in the non-degeneracy assumption.

By definition of u_i^r we have

$$\begin{aligned} u_i^r &\geq \mathbb{E}\left[-\int_0^{+\infty} e^{-rt} L\left(X_t^{0,i,\lambda}, \left(\lambda\left(X_t^{0,i,\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda}\right)}\right) dt\right] \\ &\geq \int_0^{+\infty} e^{-rt} \inf_k -L\left(k, (\lambda(k,j))_{j\in\mathcal{V}(k)}\right) dt \\ &\geq \frac{1}{r} \inf_k -L\left(k, (\lambda(k,j))_{j\in\mathcal{V}(k)}\right). \end{aligned}$$

A first lemma to study $r \to 0$

A first lemma to study $r \to 0$

Proof.

From the (lower) boundedness of the functions $(L(i,\cdot))_{i\in\mathcal{I}}$, we also have for all $(\lambda(i,j))_{i\in\mathcal{I},j\in\mathcal{V}(i)}$ that

$$\begin{aligned} & \mathbb{E}\left[-\int_0^{+\infty} e^{-rt} L\left(X_t^{0,i,\lambda}, \left(\lambda\left(X_t^{0,i,\lambda},j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda}\right)}\right) dt\right] \\ \leq \quad & -\underline{C}\int_0^{+\infty} e^{-rt} dt = -\frac{\underline{C}}{r}. \end{aligned}$$

Therefore, $u_i^r \leq -\frac{\underline{C}}{r}$.

A first lemma to study $r \to 0$

Proof.

From the (lower) boundedness of the functions $(L(i,\cdot))_{i\in\mathcal{I}}$, we also have for all $(\lambda(i,j))_{i\in\mathcal{I},j\in\mathcal{V}(i)}$ that

$$\mathbb{E}\left[-\int_0^{+\infty} e^{-rt} L\left(X_t^{0,i,\lambda}, \left(\lambda\left(X_t^{0,i,\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda}\right)}\right) dt\right]$$
$$\leq \quad -\underline{C}\int_0^{+\infty} e^{-rt} dt = -\frac{\underline{C}}{r}.$$

Therefore, $u_i^r \leq -\frac{\underline{C}}{r}$.

We conclude that $r \mapsto r u_i^r$ is bounded.

A first lemma to study $r \to 0$

A first lemma to study $r \to 0$

Proof.

Take a family of positive intensities $(\lambda(i,j))_{i\in\mathcal{I},j\in\mathcal{V}(i)}$ as in the non-degeneracy assumption.

A first lemma to study $r \to 0$

Proof.

Take a family of positive intensities $(\lambda(i,j))_{i\in\mathcal{I},j\in\mathcal{V}(i)}$ as in the non-degeneracy assumption.

Because the finite graph is connected, for all $(i,j)\in\mathcal{I}^2$ the stopping time defined by $\tau^{ij} = \inf\left\{t>0 \middle| X_t^{0,i,\lambda} = j\right\}$ verifies $\mathbb{E}\left[\tau^{ij}\right] < +\infty$.

A first lemma to study $r \to 0$

Proof.

Take a family of positive intensities $(\lambda(i,j))_{i\in\mathcal{I}, j\in\mathcal{V}(i)}$ as in the non-degeneracy assumption.

Because the finite graph is connected, for all $(i,j) \in \mathcal{I}^2$ the stopping time defined by $\tau^{ij} = \inf\left\{t > 0 \middle| X_t^{0,i,\lambda} = j\right\}$ verifies $\mathbb{E}\left[\tau^{ij}\right] < +\infty$.

So $\forall (i,j) \in \mathcal{I}^2$, we have

$$
\begin{aligned}
& u_i^r + \frac{\underline{C}}{r} \geq \mathbb{E}\left[\int_0^{\tau^{ij}} e^{-rt}\left(-L\left(X_t^{0,i,\lambda}, \left(\lambda\left(X_t^{0,i,\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda}\right)}\right) + \underline{C}\right) dt \right.\\
& \left. + e^{-r\tau^{ij}}\left(u_j^r + \frac{\underline{C}}{r}\right)\right] \\
\geq \quad & \mathbb{E}\left[\int_0^{\tau^{ij}} e^{-rt} dt\right]\left(\inf_k -L\left(k, (\lambda(k,j))_{j\in\mathcal{V}(k)}\right) + \underline{C}\right) + \mathbb{E}\left[e^{-r\tau^{ij}}\right]\left(u_j^r + \frac{\underline{C}}{r}\right) \\
\geq \quad & \mathbb{E}\left[\tau^{ij}\right]\left(\inf_k -L\left(k, (\lambda(k,j))_{j\in\mathcal{V}(k)}\right) + \underline{C}\right) + u_j^r + \frac{\underline{C}}{r}.
\end{aligned}
$$

A first lemma to study $r \to 0$

Proof.

Take a family of positive intensities $(\lambda(i,j))_{i\in\mathcal{I}, j\in\mathcal{V}(i)}$ as in the non-degeneracy assumption.

Because the finite graph is connected, for all $(i,j) \in \mathcal{I}^2$ the stopping time defined by $\tau^{ij} = \inf\left\{t > 0 \middle| X_t^{0,i,\lambda} = j\right\}$ verifies $\mathbb{E}\left[\tau^{ij}\right] < +\infty$.

So $\forall (i,j) \in \mathcal{I}^2$, we have

$$
\begin{aligned}
& u_i^r + \frac{\underline{C}}{r} \geq \mathbb{E}\left[\int_0^{\tau^{ij}} e^{-rt}\left(-L\left(X_t^{0,i,\lambda}, \left(\lambda\left(X_t^{0,i,\lambda}, j\right)\right)_{j\in\mathcal{V}\left(X_t^{0,i,\lambda}\right)}\right) + \underline{C}\right) dt \right.\\
& \left. + e^{-r\tau^{ij}}\left(u_j^r + \frac{\underline{C}}{r}\right)\right] \\
\geq \quad & \mathbb{E}\left[\int_0^{\tau^{ij}} e^{-rt} dt\right]\left(\inf_k -L\left(k, (\lambda(k,j))_{j\in\mathcal{V}(k)}\right) + \underline{C}\right) + \mathbb{E}\left[e^{-r\tau^{ij}}\right]\left(u_j^r + \frac{\underline{C}}{r}\right) \\
\geq \quad & \mathbb{E}\left[\tau^{ij}\right]\left(\inf_k -L\left(k, (\lambda(k,j))_{j\in\mathcal{V}(k)}\right) + \underline{C}\right) + u_j^r + \frac{\underline{C}}{r}.
\end{aligned}
$$

$$
\text{So } u_j^r - u_i^r \leq -\mathbb{E}\left[\tau^{ij}\right]\left(\inf_k -L\left(k, (\lambda(k,j))_{j\in\mathcal{V}(k)}\right) + \underline{C}\right).
$$

□

A second lemma to study $r \to 0$

A second lemma to study $r \to 0$

We now come to a comparison principle:

A second lemma to study $r \to 0$

We now come to a comparison principle:

Lemma

Let $\varepsilon > 0$. *Let* $(v_i)_{i\in\mathcal{I}}$ *and* $(w_i)_{i\in\mathcal{I}}$ *be such that*

$$-\varepsilon v_i + H\left(i, (v_j - v_i)_{j\in\mathcal{V}(i)}\right) \geq -\varepsilon w_i + H\left(i, (w_j - w_i)_{j\in\mathcal{V}(i)}\right), \quad \forall i \in \mathcal{I}.$$

Then $\forall i \in \mathcal{I}, v_i \leq w_i$.

A second lemma to study $r \to 0$

A second lemma to study $r \to 0$

Proof.

Let us consider $(z_i)_{i \in \mathcal{I}} = (v_i - w_i)_{i \in \mathcal{I}}$.

A second lemma to study $r \to 0$

Proof.

Let us consider $(z_i)_{i\in\mathcal{I}} = (v_i - w_i)_{i\in\mathcal{I}}$.

Let us choose $i^* \in \mathcal{I}$ such that $z_{i^*} = \max_{i\in\mathcal{I}} z_i$.

A second lemma to study $r \to 0$

Proof.

Let us consider $(z_i)_{i\in\mathcal{I}} = (v_i - w_i)_{i\in\mathcal{I}}$.
Let us choose $i^* \in \mathcal{I}$ such that $z_{i^*} = \max_{i\in\mathcal{I}} z_i$.
By definition of i^*, we know that

$$\forall j \in \mathcal{V}(i^*), v_{i^*} - w_{i^*} \geq v_j - w_j$$

i.e.

$$\forall j \in \mathcal{V}(i^*), v_j - v_{i^*} \leq w_j - w_{i^*}$$

A second lemma to study $r \to 0$

Proof.

Let us consider $(z_i)_{i\in\mathcal{I}} = (v_i - w_i)_{i\in\mathcal{I}}$.
Let us choose $i^* \in \mathcal{I}$ such that $z_{i^*} = \max_{i\in\mathcal{I}} z_i$.
By definition of i^*, we know that

$$\forall j \in \mathcal{V}(i^*), v_{i^*} - w_{i^*} \geq v_j - w_j$$

i.e.

$$\forall j \in \mathcal{V}(i^*), v_j - v_{i^*} \leq w_j - w_{i^*}$$

Because $H(i^*, \cdot)$ is nondecreasing

$$H\left(i^*, (v_j - v_{i^*})_{j\in\mathcal{V}(i^*)}\right) \leq H\left(i^*, (w_j - w_{i^*})_{j\in\mathcal{V}(i^*)}\right).$$

A second lemma to study $r \to 0$

Proof.

Let us consider $(z_i)_{i\in\mathcal{I}} = (v_i - w_i)_{i\in\mathcal{I}}$.
Let us choose $i^* \in \mathcal{I}$ such that $z_{i^*} = \max_{i\in\mathcal{I}} z_i$.
By definition of i^*, we know that

$$\forall j \in \mathcal{V}(i^*), v_{i^*} - w_{i^*} \geq v_j - w_j$$

i.e.

$$\forall j \in \mathcal{V}(i^*), v_j - v_{i^*} \leq w_j - w_{i^*}$$

Because $H(i^*, \cdot)$ is nondecreasing

$$H\left(i^*, (v_j - v_{i^*})_{j\in\mathcal{V}(i^*)}\right) \leq H\left(i^*, (w_j - w_{i^*})_{j\in\mathcal{V}(i^*)}\right).$$

We have therefore $\varepsilon(v_{i^*} - w_{i^*}) \leq 0$, so

$$\forall i \in \mathcal{I}, v_i - w_i \leq v_{i^*} - w_{i^*} \leq 0.$$

□

A third lemma to study $r \to 0$

A third lemma to study $r \to 0$

The last lemma to prove the result is:

A third lemma to study $r \to 0$

The last lemma to prove the result is:

Lemma

Let $\eta, \mu \in \mathbb{R}$. Let $(v_i)_{i\in\mathcal{I}}$ and $(w_i)_{i\in\mathcal{I}}$ be such that

$$-\eta + H\left(i, (v_j - v_i)_{j\in\mathcal{V}(i)}\right) = 0, \quad \forall i \in \mathcal{I},$$
$$-\mu + H\left(i, (w_j - w_i)_{j\in\mathcal{V}(i)}\right) = 0, \quad \forall i \in \mathcal{I}.$$

Then $\eta = \mu$.

A third lemma to study $r \to 0$

A third lemma to study $r \to 0$

Proof.

By contradiction, we can assume $\eta > \mu$ (up to an exchange).

A third lemma to study $r \to 0$

Proof.

By contradiction, we can assume $\eta > \mu$ (up to an exchange).
Let

$$C = \sup_{i \in \mathcal{I}} (w_i - v_i) + 1$$

and

$$\varepsilon = \frac{\eta - \mu}{\sup_{i \in \mathcal{I}}(w_i - v_i) - \inf_{i \in \mathcal{I}}(w_i - v_i) + 1} = \frac{\eta - \mu}{C + \sup_{i \in \mathcal{I}}(v_i - w_i)}.$$

A third lemma to study $r \to 0$

Proof.

By contradiction, we can assume $\eta > \mu$ (up to an exchange).

Let

$$C = \sup_{i \in \mathcal{I}} (w_i - v_i) + 1$$

and

$$\varepsilon = \frac{\eta - \mu}{\sup_{i \in \mathcal{I}} (w_i - v_i) - \inf_{i \in \mathcal{I}} (w_i - v_i) + 1} = \frac{\eta - \mu}{C + \sup_{i \in \mathcal{I}} (v_i - w_i)}.$$

From these definitions, we have

$$\forall i \in \mathcal{I}, \quad v_i + C > w_i \quad \text{and} \quad 0 \leq \varepsilon (v_i - w_i + C) \leq \eta - \mu.$$

A third lemma to study $r \to 0$

Proof.

By contradiction, we can assume $\eta > \mu$ (up to an exchange).

Let

$$C = \sup_{i \in \mathcal{I}} (w_i - v_i) + 1$$

and

$$\varepsilon = \frac{\eta - \mu}{\sup_{i \in \mathcal{I}}(w_i - v_i) - \inf_{i \in \mathcal{I}}(w_i - v_i) + 1} = \frac{\eta - \mu}{C + \sup_{i \in \mathcal{I}}(v_i - w_i)}.$$

From these definitions, we have

$$\forall i \in \mathcal{I}, \quad v_i + C > w_i \quad \text{and} \quad 0 \leq \varepsilon(v_i - w_i + C) \leq \eta - \mu.$$

We obtain

$$\varepsilon(v_i - w_i + C) \leq H\left(i, (v_j - v_i)_{j \in \mathcal{V}(i)}\right) - H\left(i, (w_j - w_i)_{j \in \mathcal{V}(i)}\right)$$

A third lemma to study $r \to 0$

A third lemma to study $r \to 0$

Proof.

Reorganizing the terms, we have

$$-\varepsilon w_i + H\left(i, (w_j - w_i)_{j\in\mathcal{V}(i)}\right) \leq -\varepsilon(v_i + C) + H\left(i, ((v_j + C) - (v_i + C))_{j\in\mathcal{V}(i)}\right).$$

A third lemma to study $r \to 0$

Proof.

Reorganizing the terms, we have

$$-\varepsilon w_i + H\left(i, (w_j - w_i)_{j\in\mathcal{V}(i)}\right) \leq -\varepsilon(v_i + C) + H\left(i, ((v_j + C) - (v_i + C))_{j\in\mathcal{V}(i)}\right).$$

From the previous lemma it follows that $\forall i \in \mathcal{I}, v_i + C \leq w_i$, in contradiction with the definition of C.

A third lemma to study $r \to 0$

Proof.

Reorganizing the terms, we have

$$-\varepsilon w_i + H\left(i, (w_j - w_i)_{j \in \mathcal{V}(i)}\right) \leq -\varepsilon(v_i + C) + H\left(i, ((v_j + C) - (v_i + C))_{j \in \mathcal{V}(i)}\right).$$

From the previous lemma it follows that $\forall i \in \mathcal{I}, v_i + C \leq w_i$, in contradiction with the definition of C.

We conclude $\eta = \mu$.

□

What happens when $r \to 0$

What happens when $r \to 0$

We are now ready to prove our proposition:

Proposition

We have:

- $\exists \gamma \in \mathbb{R}, \forall i \in \mathcal{I}, \lim_{r \to 0} r u_i^r = \gamma$.
- *There exists a sequence* $(r_n)_{n \in \mathbb{N}}$ *converging towards* 0 *such that* $\forall i \in \mathcal{I}, (u_i^{r_n} - u_1^{r_n})_{n \in \mathbb{N}}$ *is convergent.*
- *For all* $i \in \mathcal{I}$, *if* $\xi_i = \lim_{n \to +\infty} u_i^{r_n} - u_1^{r_n}$, *then we have*

$$-\gamma + H\left(i, (\xi_j - \xi_i)_{j \in \mathcal{V}(i)}\right) = 0.$$

Proof of what happens when $r \to 0$

Proof of what happens when $r \to 0$

Proof.

From the first lemma, we can consider a sequence $(r_n)_{n\in\mathbb{N}}$ converging towards 0, such that

$$r_n u_i^{r_n} \to \gamma_i$$

and

$$u_i^{r_n} - u_1^{r_n} \to \xi_i.$$

Proof of what happens when $r \to 0$

Proof.

From the first lemma, we can consider a sequence $(r_n)_{n\in\mathbb{N}}$ converging towards 0, such that

$$r_n u_i^{r_n} \to \gamma_i$$

and

$$u_i^{r_n} - u_1^{r_n} \to \xi_i.$$

We have

$$0 = \lim_{n\to+\infty} r_n(u_i^{r_n} - u_1^{r_n}) = \lim_{n\to+\infty} r_n u_i^{r_n} - \lim_{n\to+\infty} r_n u_1^{r_n} = \gamma_i - \gamma_1.$$

Proof of what happens when $r \to 0$

Proof.

From the first lemma, we can consider a sequence $(r_n)_{n\in\mathbb{N}}$ converging towards 0, such that

$$r_n u_i^{r_n} \to \gamma_i$$

and

$$u_i^{r_n} - u_1^{r_n} \to \xi_i.$$

We have

$$0 = \lim_{n\to+\infty} r_n(u_i^{r_n} - u_1^{r_n}) = \lim_{n\to+\infty} r_n u_i^{r_n} - \lim_{n\to+\infty} r_n u_1^{r_n} = \gamma_i - \gamma_1.$$

Therefore, $\gamma_i = \gamma$ is independent of i.

Proof of what happens when $r \to 0$

Proof of what happens when $r \to 0$

Proof.

Passing to the limit when $n \to +\infty$ in

$$-r_n u_i^{r_n} + H\left(i, \left(u_j^{r_n} - u_i^{r_n}\right)_{j \in \mathcal{V}(i)}\right) = 0$$

Proof of what happens when $r \to 0$

Proof.

Passing to the limit when $n \to +\infty$ in

$$-r_n u_i^{r_n} + H\left(i, \left(u_j^{r_n} - u_i^{r_n}\right)_{j \in \mathcal{V}(i)}\right) = 0$$

we obtain

$$-\gamma + H\left(i, (\xi_j - \xi_i)_{j \in \mathcal{V}(i)}\right) = 0.$$

Proof of what happens when $r \to 0$

Proof.

Passing to the limit when $n \to +\infty$ in

$$-r_n u_i^{r_n} + H\left(i, \left(u_j^{r_n} - u_i^{r_n}\right)_{j\in\mathcal{V}(i)}\right) = 0$$

we obtain

$$-\gamma + H\left(i, (\xi_j - \xi_i)_{j\in\mathcal{V}(i)}\right) = 0.$$

To complete the proof, we need to prove that γ is independent of the choice of the sequence $(r_n)_{n\in\mathbb{N}}$: this is a consequence of third lemma. $\square$

Comments on the limit case $r \to 0$

Comments on the limit case $r \to 0$

- The equation

$$-\gamma + H\left(i, (\xi_j - \xi_i)_{j\in\mathcal{V}(i)}\right) = 0$$

is central in the study of the limit $T \to +\infty$ when $r = 0$.

Comments on the limit case $r \to 0$

- The equation

$$-\gamma + H\left(i, (\xi_j - \xi_i)_{j \in \mathcal{V}(i)}\right) = 0$$

  is central in the study of the limit $T \to +\infty$ when $r = 0$.
- In the above equation, γ is unique (third lemma).

Comments on the limit case $r \to 0$

- The equation

$$-\gamma + H\left(i, (\xi_j - \xi_i)_{j \in \mathcal{V}(i)}\right) = 0$$

  is central in the study of the limit $T \to +\infty$ when $r = 0$.
- In the above equation, γ is unique (third lemma).
- Under some additional assumptions $(\xi_i)_i$ can be unique up a constant.

When the Hamiltonian functions are increasing

When the Hamiltonian functions are increasing

Proposition

Assume that $\forall i \in \mathcal{I}, H(i, \cdot)$ *is increasing with respect to each coordinate. Let* $(v_i)_{i\in\mathcal{I}}$ *and* $(w_i)_{i\in\mathcal{I}}$ *be such that*

$$-\gamma + H\left(i, (v_j - v_i)_{j\in\mathcal{V}(i)}\right) = 0, \quad \forall i \in \mathcal{I},$$
$$-\gamma + H\left(i, (w_j - w_i)_{j\in\mathcal{V}(i)}\right) = 0, \quad \forall i \in \mathcal{I}.$$

Then $\exists C, \forall i \in \mathcal{I}, w_i = v_i + C$, *i.e. uniqueness is true up to a constant.*

When the Hamiltonian functions are increasing

When the Hamiltonian functions are increasing

Proof.

Let us consider $C = \sup_{i \in \mathcal{I}} w_i - v_i$.

When the Hamiltonian functions are increasing

Proof.

Let us consider $C = \sup_{i \in \mathcal{I}} w_i - v_i$.

By contradiction, assume there exists $j \in \mathcal{I}$ such that $v_j + C > w_j$.

When the Hamiltonian functions are increasing

Proof.

Let us consider $C = \sup_{i \in \mathcal{I}} w_i - v_i$.

By contradiction, assume there exists $j \in \mathcal{I}$ such that $v_j + C > w_j$.

Because the graph is connected, we can find $i^* \in \mathcal{I}$ such that $v_{i^*} + C = w_{i^*}$ and such that there exists $j^* \in \mathcal{V}(i^*)$ satisfying $v_{j^*} + C > w_{j^*}$.

When the Hamiltonian functions are increasing

Proof.

Let us consider $C = \sup_{i \in \mathcal{I}} w_i - v_i$.

By contradiction, assume there exists $j \in \mathcal{I}$ such that $v_j + C > w_j$.

Because the graph is connected, we can find $i^* \in \mathcal{I}$ such that $v_{i^*} + C = w_{i^*}$ and such that there exists $j^* \in \mathcal{V}(i^*)$ satisfying $v_{j^*} + C > w_{j^*}$.

The strict monotonicity of the Hamiltonian functions implies that

$$H\left(i^*, ((v_j + C) - (v_{i^*} + C))_{j \in \mathcal{V}(i^*)}\right) > H\left(i, (w_j - w_{i^*})_{j \in \mathcal{V}(i^*)}\right)$$

in contradiction with the definition of $(v_i)_{i \in \mathcal{I}}$ and $(w_i)_{i \in \mathcal{I}}$.

When the Hamiltonian functions are increasing

Proof.

Let us consider $C = \sup_{i \in \mathcal{I}} w_i - v_i$.

By contradiction, assume there exists $j \in \mathcal{I}$ such that $v_j + C > w_j$.

Because the graph is connected, we can find $i^* \in \mathcal{I}$ such that $v_{i^*} + C = w_{i^*}$ and such that there exists $j^* \in \mathcal{V}(i^*)$ satisfying $v_{j^*} + C > w_{j^*}$.

The strict monotonicity of the Hamiltonian functions implies that

$$H\left(i^*, ((v_j + C) - (v_{i^*} + C))_{j \in \mathcal{V}(i^*)}\right) > H\left(i, (w_j - w_{i^*})_{j \in \mathcal{V}(i^*)}\right)$$

in contradiction with the definition of $(v_i)_{i \in \mathcal{I}}$ and $(w_i)_{i \in \mathcal{I}}$.

Therefore $\forall i \in \mathcal{I}, w_i = v_i + C$.

□

A general theory for optimal control on graphs – Asymptotics when $r = 0$

A change of variables

A change of variables

- Compared to the case $r > 0$, the case $r = 0$ is more subtle and more complex.

A change of variables

- Compared to the case $r > 0$, the case $r = 0$ is more subtle and more complex.
- $u_i^{T,0}(0)$ is not indeed the right "object", but rather $u_i^{T,0}(0) - \gamma T$ that will converge towards a finite limit

A change of variables

- Compared to the case $r > 0$, the case $r = 0$ is more subtle and more complex.
- $u_i^{T,0}(0)$ is not indeed the right "object", but rather $u_i^{T,0}(0) - \gamma T$ that will converge towards a finite limit $\rightarrow \gamma$ will appear to be the average gain per unit of time.

A change of variables

- Compared to the case $r > 0$, the case $r = 0$ is more subtle and more complex.
- $u_i^{T,0}(0)$ is not indeed the right "object", but rather $u_i^{T,0}(0) - \gamma T$ that will converge towards a finite limit $\rightarrow \gamma$ will appear to be the average gain per unit of time.
- To study the problem, we consider a change of variables:

$$\forall i \in \mathcal{I}, U_i : t \in \mathbb{R}_+^* \mapsto u_i^{T,0}(T - t)$$

A change of variables

- Compared to the case $r > 0$, the case $r = 0$ is more subtle and more complex.
- $u_i^{T,0}(0)$ is not indeed the right "object", but rather $u_i^{T,0}(0) - \gamma T$ that will converge towards a finite limit $\to \gamma$ will appear to be the average gain per unit of time.
- To study the problem, we consider a change of variables:

$$\forall i \in \mathcal{I}, U_i : t \in \mathbb{R}_+^* \mapsto u_i^{T,0}(T - t)$$

This function solves

$$-\frac{d}{dt} U_i(t) + H\left(i, (U_j(t) - U_i(t))_{j \in \mathcal{V}(i)}\right) = 0, \quad \forall (i,t) \in \mathcal{I} \times \mathbb{R}_+$$

with $\forall i \in \mathcal{I}, \quad U_i(0) = g(i)$.

Towards convergence

Towards convergence

For any constant C, let us introduce

$$w^C : (i,t) \in \mathcal{I} \times [0,+\infty) \mapsto w_i^C(t) = \gamma t + \xi_i + C$$

Towards convergence

For any constant C, let us introduce

$$w^C : (i, t) \in \mathcal{I} \times [0, +\infty) \mapsto w_i^C(t) = \gamma t + \xi_i + C$$

We have

$$\begin{aligned} & -\frac{d}{dt} w_i^C(t) + H\left(i, \left(w_j^C(t) - w_i^C(t)\right)_{j \in \mathcal{V}(i)}\right) \\ = \;& -\gamma + H\left(i, (\xi_j - \xi_i)_{j \in \mathcal{V}(i)}\right) \\ = \;& 0 \end{aligned}$$

Towards convergence

Towards convergence

The ODEs for U satisfy a comparison priciple similar to that proved earlier.

Towards convergence

The ODEs for U satisfy a comparison priciple similar to that proved earlier.

We can build a lower bound w^{C_1} and an upper bound w^{C_2} by:

Towards convergence

The ODEs for U satisfy a comparison priciple similar to that proved earlier.

We can build a lower bound w^{C_1} and an upper bound w^{C_2} by:

$$w_i^{C_1}(t) = \gamma t + \xi_i + C_1 \text{ with } C_1 = \min_j(g(j) - \xi_j)$$
$$w_i^{C_2}(t) = \gamma t + \xi_i + C_2 \text{ with } C_2 = \max_j(g(j) - \xi_j)$$

Towards convergence

The ODEs for U satisfy a comparison priciple similar to that proved earlier.

We can build a lower bound w^{C_1} and an upper bound w^{C_2} by:

$$w_i^{C_1}(t) = \gamma t + \xi_i + C_1 \text{ with } C_1 = \min_j(g(j) - \xi_j)$$
$$w_i^{C_2}(t) = \gamma t + \xi_i + C_2 \text{ with } C_2 = \max_j(g(j) - \xi_j)$$

We deduce that $\hat{v} : t \in [0, +\infty) \mapsto U(t) - \gamma t\vec{1}$ is bounded
$\rightarrow$ Our goal is to show that it converges when $t \rightarrow +\infty$ **under the assumption of strict monotonicity for** H.

A slightly modified equation and its properties

A slightly modified equation and its properties

$\hat{v}$ solves the slightly modified equation

$$-\frac{d}{dt}\hat{v}_i(t) - \gamma + H\left(i, (\hat{v}_j(t) - \hat{v}_i(t))_{j\in\mathcal{V}(i)}\right) = 0, \quad \forall (i,t) \in \mathcal{I} \times \mathbb{R}_+$$

with $\forall i \in \mathcal{I}, \quad \hat{v}_i(0) = g(i).$

A slightly modified equation and its properties

$\hat{v}$ solves the slightly modified equation

$$-\frac{d}{dt}\hat{v}_i(t) - \gamma + H\left(i, (\hat{v}_j(t) - \hat{v}_i(t))_{j\in\mathcal{V}(i)}\right) = 0, \quad \forall (i,t) \in \mathcal{I} \times \mathbb{R}_+$$

with $\forall i \in \mathcal{I}, \quad \hat{v}_i(0) = g(i)$.

We introduce for all $(s, y) \in \mathbb{R}_+ \times \mathbb{R}^N$ the equation

$$-\frac{d}{dt}\hat{y}_i(t) - \gamma + H\left(i, (\hat{y}_j(t) - \hat{y}_i(t))_{j\in\mathcal{V}(i)}\right) = 0, \forall (i,t) \in \mathcal{I} \times [s, +\infty), \qquad (E_{s,y})$$

with $\hat{y}_i(s) = y_i, \forall i \in \mathcal{I}$.

First property: comparison principle

First property: comparison principle

Proposition (Comparison principle)

Let $s \in \mathbb{R}_+$. Let $(\underline{y}_i)_{i\in\mathcal{I}}$ and $(\overline{y}_i)_{i\in\mathcal{I}}$ be two continuously differentiable functions on $[s, +\infty)$ such that

$$-\frac{d}{dt}\underline{y}_i(t) - \gamma + H\left(i, \left(\underline{y}_j(t) - \underline{y}_i(t)\right)_{j\in\mathcal{V}(i)}\right) \geq 0, \quad \forall (i,t) \in \mathcal{I} \times [s, +\infty),$$

$$-\frac{d}{dt}\overline{y}_i(t) - \gamma + H\left(i, \left(\overline{y}_j(t) - \overline{y}_i(t)\right)_{j\in\mathcal{V}(i)}\right) \leq 0, \quad \forall (i,t) \in \mathcal{I} \times [s, +\infty),$$

and $\forall i \in \mathcal{I}, \underline{y}_i(s) \leq \overline{y}_i(s)$.

Then $\underline{y}_i(t) \leq \overline{y}_i(t), \forall (i,t) \in \mathcal{I} \times [s, +\infty)$.

Second property: strong maximum principle

Second property: strong maximum principle

Proposition (Strong maximum principle)

Let $s \in \mathbb{R}_+$. Let $(\underline{y}_i)_{i\in\mathcal{I}}$ and $(\overline{y}_i)_{i\in\mathcal{I}}$ be two continuously differentiable functions on $[s, +\infty)$ such that

$$-\frac{d}{dt}\underline{y}_i(t) - \gamma + H\left(i, \left(\underline{y}_j(t) - \underline{y}_i(t)\right)_{j\in\mathcal{V}(i)}\right) = 0, \quad \forall (i,t) \in \mathcal{I} \times [s, +\infty),$$

$$-\frac{d}{dt}\overline{y}_i(t) - \gamma + H\left(i, \left(\overline{y}_j(t) - \overline{y}_i(t)\right)_{j\in\mathcal{V}(i)}\right) = 0, \quad \forall (i,t) \in \mathcal{I} \times [s, +\infty),$$

and $\underline{y}(s) \lneq \overline{y}(s)$, i.e. $\forall j \in \mathcal{I}, \underline{y}_j(s) \leq \overline{y}_j(s)$ and $\exists i \in \mathcal{I}, \underline{y}_i(s) < \overline{y}_i(s)$.

Then $\underline{y}_i(t) < \overline{y}_i(t), \forall (i,t) \in \mathcal{I} \times (s, +\infty)$.

Second property: strong maximum principle

Second property: strong maximum principle

Proof.

If there exists $(i, \bar{t}) \in \mathcal{I} \times (s, +\infty)$ such that $\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t})$, then $\bar{t}$ is a maximizer of the function $t \in (s, +\infty) \mapsto \underline{y}_i(t) - \overline{y}_i(t)$. Hence, $\frac{d}{dt}\underline{y}_i(\bar{t}) = \frac{d}{dt}\overline{y}_i(\bar{t})$.

Second property: strong maximum principle

Proof.

If there exists $(i, \bar{t}) \in \mathcal{I} \times (s, +\infty)$ such that $\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t})$, then $\bar{t}$ is a maximizer of the function $t \in (s, +\infty) \mapsto \underline{y}_i(t) - \overline{y}_i(t)$. Hence, $\frac{d}{dt}\underline{y}_i(\bar{t}) = \frac{d}{dt}\overline{y}_i(\bar{t})$.

We deduce that

$$\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t}) \implies H\left(i, \left(\underline{y}_j(\bar{t}) - \underline{y}_i(\bar{t})\right)_{j \in \mathcal{V}(i)}\right) = H\left(i, \left(\overline{y}_j(\bar{t}) - \overline{y}_i(\bar{t})\right)_{j \in \mathcal{V}(i)}\right)$$

Second property: strong maximum principle

Proof.

If there exists $(i, \bar{t}) \in \mathcal{I} \times (s, +\infty)$ such that $\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t})$, then $\bar{t}$ is a maximizer of the function $t \in (s, +\infty) \mapsto \underline{y}_i(t) - \overline{y}_i(t)$. Hence, $\frac{d}{dt}\underline{y}_i(\bar{t}) = \frac{d}{dt}\overline{y}_i(\bar{t})$.

We deduce that

$$\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t}) \implies H\left(i, \left(\underline{y}_j(\bar{t}) - \underline{y}_i(\bar{t})\right)_{j \in \mathcal{V}(i)}\right) = H\left(i, \left(\overline{y}_j(\bar{t}) - \overline{y}_i(\bar{t})\right)_{j \in \mathcal{V}(i)}\right)$$

Because $H(i, \cdot)$ is increasing,

$$\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t}) \implies \forall j \in \mathcal{V}(i), \underline{y}_j(\bar{t}) = \overline{y}_j(\bar{t})$$

Second property: strong maximum principle

Proof.

If there exists $(i, \bar{t}) \in \mathcal{I} \times (s, +\infty)$ such that $\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t})$, then $\bar{t}$ is a maximizer of the function $t \in (s, +\infty) \mapsto \underline{y}_i(t) - \overline{y}_i(t)$. Hence, $\frac{d}{dt}\underline{y}_i(\bar{t}) = \frac{d}{dt}\overline{y}_i(\bar{t})$.

We deduce that

$$\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t}) \implies H\left(i, \left(\underline{y}_j(\bar{t}) - \underline{y}_i(\bar{t})\right)_{j \in \mathcal{V}(i)}\right) = H\left(i, \left(\overline{y}_j(\bar{t}) - \overline{y}_i(\bar{t})\right)_{j \in \mathcal{V}(i)}\right)$$

Because $H(i, \cdot)$ is increasing,

$$\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t}) \implies \forall j \in \mathcal{V}(i), \underline{y}_j(\bar{t}) = \overline{y}_j(\bar{t})$$

As the graph is connected,

$$\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t}) \implies \forall j \in \mathcal{I}, \underline{y}_j(\bar{t}) = \overline{y}_j(\bar{t})$$

Second property: strong maximum principle

Second property: strong maximum principle

Proof.

If there exists $(i, \bar{t}) \in \mathcal{I} \times (s, +\infty)$ such that $\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t})$, we define

$$F = \left\{ t \in (s, +\infty), \forall j \in \mathcal{I}, \underline{y}_j(t) = \overline{y}_j(t) \right\}.$$

Second property: strong maximum principle

Proof.

If there exists $(i, \bar{t}) \in \mathcal{I} \times (s, +\infty)$ such that $\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t})$, we define

$$F = \left\{ t \in (s, +\infty), \forall j \in \mathcal{I}, \underline{y}_j(t) = \overline{y}_j(t) \right\}.$$

We have:

Second property: strong maximum principle

Proof.

If there exists $(i, \bar{t}) \in \mathcal{I} \times (s, +\infty)$ such that $\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t})$, we define

$$F = \left\{ t \in (s, +\infty), \forall j \in \mathcal{I}, \underline{y}_j(t) = \overline{y}_j(t) \right\}.$$

We have:

- F is nonempty since $\bar{t} \in F$.

Second property: strong maximum principle

Proof.

If there exists $(i, \bar{t}) \in \mathcal{I} \times (s, +\infty)$ such that $\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t})$, we define

$$F = \left\{ t \in (s, +\infty), \forall j \in \mathcal{I}, \underline{y}_j(t) = \overline{y}_j(t) \right\}.$$

We have:

- F is nonempty since $\bar{t} \in F$.
- F is also closed.

Second property: strong maximum principle

Proof.

If there exists $(i, \bar{t}) \in \mathcal{I} \times (s, +\infty)$ such that $\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t})$, we define

$$F = \left\{ t \in (s, +\infty), \forall j \in \mathcal{I}, \underline{y}_j(t) = \overline{y}_j(t) \right\}.$$

We have:

- F is nonempty since $\bar{t} \in F$.
- F is also closed.
- $\underline{y}(s) \lneq \overline{y}(s)$ implies that $t^* = \inf F = \min F > s$.

Second property: strong maximum principle

Proof.

If there exists $(i, \bar{t}) \in \mathcal{I} \times (s, +\infty)$ such that $\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t})$, we define

$$F = \left\{ t \in (s, +\infty), \forall j \in \mathcal{I}, \underline{y}_j(t) = \overline{y}_j(t) \right\}.$$

We have:

- F is nonempty since $\bar{t} \in F$.
- F is also closed.
- $\underline{y}(s) \lneq \overline{y}(s)$ implies that $t^* = \inf F = \min F > s$.

$\underline{y}$ and $\overline{y}$ are two local solutions of the Cauchy problem $(E_{t^*, \underline{y}(t^*)})$ so they are equal in a neighborhood of t^*... which contradicts the definition of t^*.

Second property: strong maximum principle

Proof.

If there exists $(i, \bar{t}) \in \mathcal{I} \times (s, +\infty)$ such that $\underline{y}_i(\bar{t}) = \overline{y}_i(\bar{t})$, we define

$$F = \left\{ t \in (s, +\infty), \forall j \in \mathcal{I}, \underline{y}_j(t) = \overline{y}_j(t) \right\}.$$

We have:

- F is nonempty since $\bar{t} \in F$.
- F is also closed.
- $\underline{y}(s) \lneq \overline{y}(s)$ implies that $t^* = \inf F = \min F > s$.

$\underline{y}$ and $\overline{y}$ are two local solutions of the Cauchy problem $(E_{t^*, \underline{y}(t^*)})$ so they are equal in a neighborhood of t^*... which contradicts the definition of t^*.

We conclude that

$$\underline{y}_i(t) < \overline{y}_i(t), \forall (i, t) \in \mathcal{I} \times (s, +\infty).$$

□

Third property: semi-group and continuity

Third property: semi-group and continuity

For all $t \in \mathbb{R}_+$, we introduce the operator $S(t) : y \in \mathbb{R}^N \mapsto \hat{y}(t) \in \mathbb{R}^N$, where $\hat{y}$ is the solution of $(E_{0,y})$.

Third property: semi-group and continuity

For all $t \in \mathbb{R}_+$, we introduce the operator $S(t) : y \in \mathbb{R}^N \mapsto \hat{y}(t) \in \mathbb{R}^N$, where $\hat{y}$ is the solution of $(E_{0,y})$.

Proposition

S satisfies the following properties:

- $\forall t, t' \in \mathbb{R}_+, S(t) \circ S(t') = S(t + t') = S(t') \circ S(t)$.
- $\forall t \in \mathbb{R}_+, \forall x, y \in \mathbb{R}^N, \|S(t)(x) - S(t)(y)\|_\infty \leq \|x - y\|_\infty$. *In particular, $S(t)$ is continuous.*

Third property: semi-group and continuity

Third property: semi-group and continuity

Proof.

The first point is trivial (Picard-Lindelöf).

Proof.

The first point is trivial (Picard-Lindelöf).

For the second point, let us introduce

$$\underline{y} : t \in \mathbb{R}_+ \mapsto S(t)(x) \quad \text{and} \quad \overline{y} : t \in \mathbb{R}_+ \mapsto S(t)(y) + \|x - y\|_\infty \vec{1}$$

Third property: semi-group and continuity

Proof.

The first point is trivial (Picard-Lindelöf).

For the second point, let us introduce

$$\underline{y} : t \in \mathbb{R}_+ \mapsto S(t)(x) \quad \text{and} \quad \overline{y} : t \in \mathbb{R}_+ \mapsto S(t)(y) + \|x - y\|_\infty \vec{1}$$

We have $\underline{y}(0) = x \leq y + \|x - y\|_\infty \vec{1} = \overline{y}(0)$, so

$$\forall t \in \mathbb{R}_+, \underline{y}(t) \leq \overline{y}(t)$$

Third property: semi-group and continuity

Proof.

The first point is trivial (Picard-Lindelöf).

For the second point, let us introduce

$$\underline{y} : t \in \mathbb{R}_+ \mapsto S(t)(x) \quad \text{and} \quad \overline{y} : t \in \mathbb{R}_+ \mapsto S(t)(y) + \|x - y\|_\infty \vec{1}$$

We have $\underline{y}(0) = x \leq y + \|x - y\|_\infty \vec{1} = \overline{y}(0)$, so

$$\forall t \in \mathbb{R}_+, \underline{y}(t) \leq \overline{y}(t)$$

i.e.

$$\forall t \in \mathbb{R}_+, \quad S(t)(x) \leq S(t)(y) + \|x - y\|_\infty \vec{1}.$$

Third property: semi-group and continuity

Proof.

The first point is trivial (Picard-Lindelöf).

For the second point, let us introduce

$$\underline{y} : t \in \mathbb{R}_+ \mapsto S(t)(x) \quad \text{and} \quad \overline{y} : t \in \mathbb{R}_+ \mapsto S(t)(y) + \|x - y\|_\infty \vec{1}$$

We have $\underline{y}(0) = x \leq y + \|x - y\|_\infty \vec{1} = \overline{y}(0)$, so

$$\forall t \in \mathbb{R}_+, \underline{y}(t) \leq \overline{y}(t)$$

i.e.

$$\forall t \in \mathbb{R}_+, \quad S(t)(x) \leq S(t)(y) + \|x - y\|_\infty \vec{1}.$$

Reversing the role of x and y we obtain

$$\|S(t)(x) - S(t)(y)\|_\infty \leq \|x - y\|_\infty .$$

□

Dynamics of the upper bound

Dynamics of the upper bound

In order to study the asymptotic behavior of $\hat{v}$, we define the function

$$q : t \in \mathbb{R}_+ \mapsto q(t) = \sup_{i \in \mathcal{I}} (\hat{v}_i(t) - \xi_i).$$

Dynamics of the upper bound

In order to study the asymptotic behavior of $\hat{v}$, we define the function

$$q : t \in \mathbb{R}_+ \mapsto q(t) = \sup_{i \in \mathcal{I}} (\hat{v}_i(t) - \xi_i).$$

We have the following lemma:

Dynamics of the upper bound

In order to study the asymptotic behavior of $\hat{v}$, we define the function

$$q : t \in \mathbb{R}_+ \mapsto q(t) = \sup_{i \in \mathcal{I}} (\hat{v}_i(t) - \xi_i).$$

We have the following lemma:

Lemma

q is a nonincreasing function, bounded from below. We denote by $q_\infty = \lim_{t \to +\infty} q(t)$ its lower bound.

Dynamics of the upper bound

Dynamics of the upper bound

Proof.

Let $s \in \mathbb{R}_+$. Let us define $\underline{y} : (i, t) \in \mathcal{I} \times [s, \infty) \mapsto \hat{v}_i(t)$ and $\overline{y} : (i, t) \in \mathcal{I} \times [s, \infty) \mapsto q(s) + \xi_i$.

Dynamics of the upper bound

Proof.

Let $s \in \mathbb{R}_+$. Let us define $\underline{y} : (i,t) \in \mathcal{I} \times [s, \infty) \mapsto \hat{v}_i(t)$ and $\overline{y} : (i,t) \in \mathcal{I} \times [s, \infty) \mapsto q(s) + \xi_i$.

We have $\forall i \in \mathcal{I}, \underline{y}_i(s) \leq \overline{y}_i(s)$ and

$$-\frac{d}{dt}\overline{y}_i(t) - \gamma + H\left(i, \left(\overline{y}_j(t) - \overline{y}_i(t)\right)_{j \in \mathcal{V}(i)}\right)$$

$$= -\gamma + H\left(i, (\xi_j - \xi_i)_{j \in \mathcal{V}(i)}\right) = 0, \forall (i,t) \in \mathcal{I} \times [s, +\infty).$$

We conclude that $\forall (i,t) \in \mathcal{I} \times [s, +\infty), \underline{y}_i(t) \leq \overline{y}_i(t)$, i.e. $\hat{v}_i(t) \leq q(s) + \xi_i$. In particular $q(t) \leq q(s), \forall t \geq s$.

Dynamics of the upper bound

Proof.

Let $s \in \mathbb{R}_+$. Let us define $\underline{y} : (i,t) \in \mathcal{I} \times [s, \infty) \mapsto \hat{v}_i(t)$ and $\overline{y} : (i,t) \in \mathcal{I} \times [s, \infty) \mapsto q(s) + \xi_i$.

We have $\forall i \in \mathcal{I}, \underline{y}_i(s) \leq \overline{y}_i(s)$ and

$$-\frac{d}{dt}\overline{y}_i(t) - \gamma + H\left(i, \left(\overline{y}_j(t) - \overline{y}_i(t)\right)_{j \in \mathcal{V}(i)}\right)$$

$$= -\gamma + H\left(i, (\xi_j - \xi_i)_{j \in \mathcal{V}(i)}\right) = 0, \forall (i,t) \in \mathcal{I} \times [s, +\infty).$$

We conclude that $\forall (i,t) \in \mathcal{I} \times [s, +\infty), \underline{y}_i(t) \leq \overline{y}_i(t)$, i.e. $\hat{v}_i(t) \leq q(s) + \xi_i$. In particular $q(t) \leq q(s), \forall t \geq s$.

Because $\hat{v}$ is bounded, so is q and its limit $q_\infty = \lim_{t \to +\infty} q(t)$.

□

The convergence theorem

The convergence theorem

Theorem

The asymptotic behavior of $\hat{v}$ is given by

$$\forall i \in \mathcal{I}, \lim_{t \to +\infty} \hat{v}_i(t) = \xi_i + q_\infty.$$

The convergence theorem

The convergence theorem

Proof.

As $\hat{v}$ is bounded, there exists $(t_n)_n$ converging towards $+\infty$ such that $\hat{v}(t_n) \to \hat{v}_\infty \leq \xi + q_\infty \vec{1}$.

The convergence theorem

Proof.

As $\hat{v}$ is bounded, there exists $(t_n)_n$ converging towards $+\infty$ such that $\hat{v}(t_n) \to \hat{v}_\infty \leq \xi + q_\infty \vec{1}$.

Because $\hat{v}$ is bounded and satisfies $(E_{0,y})$ for $y = (y_i)_{i \in \mathcal{I}} = (g(i))_{i \in \mathcal{I}}$, we can apply Arzelà–Ascoli theorem to

$$\mathcal{K} = \{s \in [0,1] \mapsto \hat{v}(t_n + s) | n \in \mathbb{N}\}.$$

Proof.

As $\hat{v}$ is bounded, there exists $(t_n)_n$ converging towards $+\infty$ such that $\hat{v}(t_n) \to \hat{v}_\infty \leq \xi + q_\infty \vec{1}$.

Because $\hat{v}$ is bounded and satisfies $(E_{0,y})$ for $y = (y_i)_{i\in\mathcal{I}} = (g(i))_{i\in\mathcal{I}}$, we can apply Arzelà–Ascoli theorem to

$$\mathcal{K} = \left\{ s \in [0,1] \mapsto \hat{v}(t_n + s) | n \in \mathbb{N} \right\}.$$

There exists a subsequence $\left(t_{\phi(n)}\right)_n$ and a function $z \in C^0\left([0,1], \mathbb{R}^N\right)$ such that $\left(s \in [0,1] \mapsto \hat{v}\left(t_{\phi(n)} + s\right)\right)_n$ converges uniformly towards z (with $z(0) = \hat{v}_\infty$).

The convergence theorem

Proof.

As $\hat{v}$ is bounded, there exists $(t_n)_n$ converging towards $+\infty$ such that $\hat{v}(t_n) \to \hat{v}_\infty \leq \xi + q_\infty \vec{1}$.

Because $\hat{v}$ is bounded and satisfies $(E_{0,y})$ for $y = (y_i)_{i\in\mathcal{I}} = (g(i))_{i\in\mathcal{I}}$, we can apply Arzelà–Ascoli theorem to

$$\mathcal{K} = \{s \in [0,1] \mapsto \hat{v}(t_n + s) | n \in \mathbb{N}\}.$$

There exists a subsequence $\left(t_{\phi(n)}\right)_n$ and a function $z \in C^0\left([0,1], \mathbb{R}^N\right)$ such that $\left(s \in [0,1] \mapsto \hat{v}\left(t_{\phi(n)} + s\right)\right)_n$ converges uniformly towards z (with $z(0) = \hat{v}_\infty$). Using the results on the semi-group, we have that z solves the ODEs:

$$\forall t \in [0,1], S(t)(z(0)) = S(t)\left(\lim_{n\to+\infty} \hat{v}\left(t_{\phi(n)}\right)\right) = \lim_{n\to+\infty} S(t)\left(\hat{v}\left(t_{\phi(n)}\right)\right)$$

$$= \lim_{n\to+\infty} \hat{v}\left(t + t_{\phi(n)}\right) = z(t).$$

The convergence theorem

The convergence theorem

Proof.

Now, if

$$z(0) = \hat{v}_\infty \lneq \xi + q_\infty \vec{1}$$

The convergence theorem

Proof.

Now, if

$$z(0) = \hat{v}_\infty \lneq \xi + q_\infty \vec{1}$$

then the strong maximum principle implies that

$$z(1) < \xi + q_\infty \vec{1}.$$

The convergence theorem

Proof.

Now, if

$$z(0) = \hat{v}_\infty \lneq \xi + q_\infty \vec{1}$$

then the strong maximum principle implies that

$$z(1) < \xi + q_\infty \vec{1}.$$

Therefore there exists $n \in \mathbb{N}$ such that $\hat{v}\left(t_{\phi(n)} + 1\right) < \xi + q_\infty \vec{1}$.

Proof.

Now, if

$$z(0) = \hat{v}_\infty \lneq \xi + q_\infty \vec{1}$$

then the strong maximum principle implies that

$$z(1) < \xi + q_\infty \vec{1}.$$

Therefore there exists $n \in \mathbb{N}$ such that $\hat{v}\left(t_{\phi(n)} + 1\right) < \xi + q_\infty \vec{1}$. This implies $q\left(t_{\phi(n)} + 1\right) < q_\infty$: a contradiction.

The convergence theorem

Proof.

Now, if

$$z(0) = \hat{v}_\infty \lneq \xi + q_\infty \vec{1}$$

then the strong maximum principle implies that

$$z(1) < \xi + q_\infty \vec{1}.$$

Therefore there exists $n \in \mathbb{N}$ such that $\hat{v}\left(t_{\phi(n)} + 1\right) < \xi + q_\infty \vec{1}$. This implies $q\left(t_{\phi(n)} + 1\right) < q_\infty$: a contradiction.

This means that $z(0) = \hat{v}_\infty = \xi + q_\infty \vec{1}$.

The convergence theorem

Proof.

Now, if

$$z(0) = \hat{v}_\infty \lneq \xi + q_\infty \vec{1}$$

then the strong maximum principle implies that

$$z(1) < \xi + q_\infty \vec{1}.$$

Therefore there exists $n \in \mathbb{N}$ such that $\hat{v}\left(t_{\phi(n)} + 1\right) < \xi + q_\infty \vec{1}$. This implies $q\left(t_{\phi(n)} + 1\right) < q_\infty$: a contradiction.

This means that $z(0) = \hat{v}_\infty = \xi + q_\infty \vec{1}$.

In other words, for any sequence $(t_n)_n$ converging towards $+\infty$ such that $(\hat{v}(t_n))_n$ is convergent, the limit is $\xi + q_\infty \vec{1}$.

Proof.

Now, if

$$z(0) = \hat{v}_\infty \lneq \xi + q_\infty \vec{1}$$

then the strong maximum principle implies that

$$z(1) < \xi + q_\infty \vec{1}.$$

Therefore there exists $n \in \mathbb{N}$ such that $\hat{v}\left(t_{\phi(n)} + 1\right) < \xi + q_\infty \vec{1}$. This implies $q\left(t_{\phi(n)} + 1\right) < q_\infty$: a contradiction.

This means that $z(0) = \hat{v}_\infty = \xi + q_\infty \vec{1}$.

In other words, for any sequence $(t_n)_n$ converging towards $+\infty$ such that $(\hat{v}(t_n))_n$ is convergent, the limit is $\xi + q_\infty \vec{1}$.

This means that $\forall i \in \mathcal{I}, \lim_{t\to+\infty} \hat{v}_i(t) = \xi_i + q_\infty$.

□

Conclusion for the optimal control problem

Conclusion for the optimal control problem

Corollary

The asymptotic behavior of the value functions associated with our problem when $r = 0$ is given by

$$\forall i \in \mathcal{I}, \forall t \in \mathbb{R}_+, u_i^{T,r}(t) = \gamma(T - t) + \xi_i + q_\infty + \underset{T \to +\infty}{o}(1).$$

The limit points of the associated optimal controls for all $t \in \mathbb{R}_+$ as $T \to +\infty$ are feedback control functions verifying $\forall i \in \mathcal{I}, \forall j \in \mathcal{V}(i)$:

$$\lambda(i,j) \in \underset{\left(\lambda_{ij}\right)_{j \in \mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|}}{argmax} \left(\left(\sum_{j \in \mathcal{V}(i)} \lambda_{ij}(\xi_j - \xi_i) \right) - L\left(i, \left(\lambda_{ij}\right)_{j \in \mathcal{V}(i)}\right) \right)$$

Conclusion for the optimal control problem

Corollary

The asymptotic behavior of the value functions associated with our problem when $r = 0$ is given by

$$\forall i \in \mathcal{I}, \forall t \in \mathbb{R}_+, u_i^{T,r}(t) = \gamma(T - t) + \xi_i + q_\infty + \underset{T \to +\infty}{o}(1).$$

The limit points of the associated optimal controls for all $t \in \mathbb{R}_+$ as $T \to +\infty$ are feedback control functions verifying $\forall i \in \mathcal{I}, \forall j \in \mathcal{V}(i)$:

$$\lambda(i,j) \in \underset{(\lambda_{ij})_{j \in \mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|}}{argmax} \left(\left(\sum_{j \in \mathcal{V}(i)} \lambda_{ij}(\xi_j - \xi_i) \right) - L\left(i, (\lambda_{ij})_{j \in \mathcal{V}(i)}\right) \right)$$

Remark: if $(L(i,\cdot))_i$ are convex functions that are strictly convex on their domain, the Hamiltonian functions $(H(i,\cdot))_i$ are differentiable and the optimal controls converge towards the unique element of the above argmax.

Conclusions about the general theory

Conclusions about the general theory

Conclusions about the general theory

What we have seen

Conclusions about the general theory

What we have seen

- We have seen that optimal control problems on graphs appear naturally.

Conclusions about the general theory

What we have seen

- We have seen that optimal control problems on graphs appear naturally.
- We have provided, under simple assumptions, a way to characterize optimal controls (with ODEs).

Conclusions about the general theory

What we have seen

- We have seen that optimal control problems on graphs appear naturally.
- We have provided, under simple assumptions, a way to characterize optimal controls (with ODEs).
- We have generalized the results to the case of infinite horizon problems when $r > 0$ (stationary problems).

Conclusions about the general theory

What we have seen

- We have seen that optimal control problems on graphs appear naturally.
- We have provided, under simple assumptions, a way to characterize optimal controls (with ODEs).
- We have generalized the results to the case of infinite horizon problems when $r > 0$ (stationary problems).
- We have obtained a (difficult) result on the asymptotic behavior far from T when $r = 0$.

Conclusions about the general theory

What we have seen

- We have seen that optimal control problems on graphs appear naturally.
- We have provided, under simple assumptions, a way to characterize optimal controls (with ODEs).
- We have generalized the results to the case of infinite horizon problems when $r > 0$ (stationary problems).
- We have obtained a (difficult) result on the asymptotic behavior far from T when $r = 0$.

What we are going to see now

Conclusions about the general theory

What we have seen

- We have seen that optimal control problems on graphs appear naturally.
- We have provided, under simple assumptions, a way to characterize optimal controls (with ODEs).
- We have generalized the results to the case of infinite horizon problems when $r > 0$ (stationary problems).
- We have obtained a (difficult) result on the asymptotic behavior far from T when $r = 0$.

What we are going to see now

- A special case where all equations can be transformed into linear ones

Conclusions about the general theory

What we have seen

- We have seen that optimal control problems on graphs appear naturally.
- We have provided, under simple assumptions, a way to characterize optimal controls (with ODEs).
- We have generalized the results to the case of infinite horizon problems when $r > 0$ (stationary problems).
- We have obtained a (difficult) result on the asymptotic behavior far from T when $r = 0$.

What we are going to see now

- A special case where all equations can be transformed into linear ones
  $\rightarrow$ Intensive use of linear algebra and matrix analysis.

Conclusions about the general theory

What we have seen

- We have seen that optimal control problems on graphs appear naturally.
- We have provided, under simple assumptions, a way to characterize optimal controls (with ODEs).
- We have generalized the results to the case of infinite horizon problems when $r > 0$ (stationary problems).
- We have obtained a (difficult) result on the asymptotic behavior far from T when $r = 0$.

What we are going to see now

- A special case where all equations can be transformed into linear ones
  $\rightarrow$ Intensive use of linear algebra and matrix analysis.
- An important application to market making: the solution to Avellaneda-Stoikov equations.

Entropic costs: when nonlinearities vanish

Introduction

Introduction

We previously considered a general framework. In what follows we consider a specific case of interest:

Introduction

We previously considered a general framework. In what follows we consider a specific case of interest:

Assumptions

Introduction

We previously considered a general framework. In what follows we consider a specific case of interest:

Assumptions

- No discount rate: $r = 0$

Introduction

We previously considered a general framework. In what follows we consider a specific case of interest:

Assumptions

- No discount rate: $r = 0$
- Functions L of the following form:

$$L(i, \cdot) : (\lambda_{ij})_{j \in \mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|} \mapsto L\left(i, (\lambda_{ij})_{j \in \mathcal{V}(i)}\right)$$

  where

$$L\left(i, (\lambda_{ij})_{j \in \mathcal{V}(i)}\right) = -h(i) + \sum_{j \in \mathcal{V}(i)} \left(\lambda_{ij} \log(\lambda_{ij}) + b_{ij}\lambda_{ij}\right)$$

Introduction

We previously considered a general framework. In what follows we consider a specific case of interest:

Assumptions

- No discount rate: $r = 0$
- Functions L of the following form:

$$L(i, \cdot) : (\lambda_{ij})_{j \in \mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|} \mapsto L\left(i, (\lambda_{ij})_{j \in \mathcal{V}(i)}\right)$$

where

$$L\left(i, (\lambda_{ij})_{j \in \mathcal{V}(i)}\right) = -h(i) + \sum_{j \in \mathcal{V}(i)} \left(\lambda_{ij} \log(\lambda_{ij}) + b_{ij}\lambda_{ij}\right)$$

- These functions L satisfy the assumptions of the previous sections.

Introduction

We previously considered a general framework. In what follows we consider a specific case of interest:

Assumptions

- No discount rate: $r = 0$
- Functions L of the following form:

$$L(i, \cdot) : (\lambda_{ij})_{j \in \mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|} \mapsto L\left(i, (\lambda_{ij})_{j \in \mathcal{V}(i)}\right)$$

where

$$L\left(i, (\lambda_{ij})_{j \in \mathcal{V}(i)}\right) = -h(i) + \sum_{j \in \mathcal{V}(i)} \left(\lambda_{ij} \log(\lambda_{ij}) + b_{ij}\lambda_{ij}\right)$$

- These functions L satisfy the assumptions of the previous sections.
- Because of the term $\sum_{j \in \mathcal{V}(i)} \lambda_{ij} \log(\lambda_{ij})$, we talk of entropic costs.

The Hamiltonian functions

The Hamiltonian functions

The interest of this family of cost functions lies in the resulting form of the Hamiltonian functions:

The Hamiltonian functions

The interest of this family of cost functions lies in the resulting form of the Hamiltonian functions:

Proposition

$\forall i, \forall p = (p_j)_{j \in \mathcal{V}(i)} \in \mathbb{R}^{|\mathcal{V}(i)|}$,

$$H(i,p) = h(i) + \sum_{j \in \mathcal{V}(i)} e^{-1-b_{ij}} e^{p_j}.$$

Moreover, the supremum in the definition of $H(i,p)$ is reached when

$$\forall j \in \mathcal{V}(i), \quad \lambda_{ij} = \lambda^*_{ij} = e^{-1-b_{ij}} e^{p_j}.$$

The Hamiltonian functions

The Hamiltonian functions

Proof.

$$H(i,p) = h(i) + \sup_{(\lambda_{ij})_{j\in\mathcal{V}(i)}\in\mathbb{R}_+^{|\mathcal{V}(i)|}} \sum_{j\in\mathcal{V}(i)} \left(\lambda_{ij} p_j - \left(\lambda_{ij}\log(\lambda_{ij}) + b_{ij}\lambda_{ij}\right)\right).$$

The Hamiltonian functions

Proof.

$$H(i,p) = h(i) + \sup_{(\lambda_{ij})_{j\in\mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|}} \sum_{j\in\mathcal{V}(i)} \left(\lambda_{ij} p_j - (\lambda_{ij}\log(\lambda_{ij}) + b_{ij}\lambda_{ij})\right).$$

The first order condition associated with the supremum writes:

The Hamiltonian functions

Proof.

$$H(i,p) = h(i) + \sup_{(\lambda_{ij})_{j\in\mathcal{V}(i)} \in \mathbb{R}_+^{|\mathcal{V}(i)|}} \sum_{j\in\mathcal{V}(i)} \left(\lambda_{ij} p_j - (\lambda_{ij}\log(\lambda_{ij}) + b_{ij}\lambda_{ij})\right).$$

The first order condition associated with the supremum writes:

$$\forall j \in \mathcal{V}(i), p_j - \log(\lambda^*_{ij}) - 1 - b_{ij} = 0$$

The Hamiltonian functions

Proof.

$$H(i,p) = h(i) + \sup_{(\lambda_{ij})_{j\in\mathcal{V}(i)}\in\mathbb{R}_+^{|\mathcal{V}(i)|}} \sum_{j\in\mathcal{V}(i)} \left(\lambda_{ij}p_j - (\lambda_{ij}\log(\lambda_{ij}) + b_{ij}\lambda_{ij})\right).$$

The first order condition associated with the supremum writes:

$$\forall j \in \mathcal{V}(i), p_j - \log(\lambda^*_{ij}) - 1 - b_{ij} = 0$$

i.e.

$$\forall j \in \mathcal{V}(i), \quad \lambda^*_{ij} = e^{-1-b_{ij}} e^{p_j}.$$

The Hamiltonian functions

Proof.

$$H(i,p) = h(i) + \sup_{(\lambda_{ij})_{j\in\mathcal{V}(i)}\in\mathbb{R}_+^{|\mathcal{V}(i)|}} \sum_{j\in\mathcal{V}(i)} \left(\lambda_{ij} p_j - (\lambda_{ij}\log(\lambda_{ij}) + b_{ij}\lambda_{ij})\right).$$

The first order condition associated with the supremum writes:

$$\forall j \in \mathcal{V}(i), p_j - \log(\lambda^*_{ij}) - 1 - b_{ij} = 0$$

i.e.

$$\forall j \in \mathcal{V}(i), \quad \lambda^*_{ij} = e^{-1-b_{ij}} e^{p_j}.$$

Plugging that formula, we obtain

$$H(i,p) = h(i) + \sum_{j\in\mathcal{V}(i)} e^{-1-b_{ij}} e^{p_j}.$$

□

Hamilton-Jacobi / Bellman equations

Hamilton-Jacobi / Bellman equations

The ODEs characterizing the value function writes:

$\forall (i,t) \in \mathcal{I} \times [0,T]$,

$$\frac{d}{dt} V_i^T(t) + H\left(i, \left(V_j^T(t) - V_i^T(t)\right)_{j \in \mathcal{V}(i)}\right) = 0$$

with terminal condition $V_i^T(T) = g(i), \quad \forall i \in \mathcal{I}$.

Hamilton-Jacobi / Bellman equations

The ODEs characterizing the value function writes:

$\forall (i,t) \in \mathcal{I} \times [0,T]$,

$$\frac{d}{dt} V_i^T(t) + H\left(i, \left(V_j^T(t) - V_i^T(t)\right)_{j \in \mathcal{V}(i)}\right) = 0$$

with terminal condition $V_i^T(T) = g(i), \quad \forall i \in \mathcal{I}$.

In the present case:

$\forall (i,t) \in \mathcal{I} \times [0,T]$,

$$\frac{d}{dt} V_i^T(t) + h(i) + \sum_{j \in \mathcal{V}(i)} e^{-1-b_{ij}} \exp\left(V_j^T(t) - V_i^T(t)\right) = 0$$

with terminal condition $V_i^T(T) = g(i), \quad \forall i \in \mathcal{I}$.

Change of variables

Change of variables

Let us introduce the change of variables

$$\forall (i,t) \in \mathcal{I} \times [0,T], w_i^T(t) = \exp\left(V_i^T(t)\right)$$

Change of variables

Let us introduce the change of variables

$$\forall (i,t) \in \mathcal{I} \times [0,T], w_i^T(t) = \exp\left(V_i^T(t)\right)$$

Then the system of ODEs writes

$\forall (i,t) \in \mathcal{I} \times [0,T]$,

$$\frac{d}{dt} w_i^T(t) + h(i) w_i^T(t) + \sum_{j \in \mathcal{V}(i)} e^{-1-b_{ij}} w_j^T(t) = 0$$

with terminal condition $w_i^T(T) = e^{g(i)}, \quad \forall i \in \mathcal{I}$.

Change of variables

Let us introduce the change of variables

$$\forall (i,t) \in \mathcal{I} \times [0,T], w_i^T(t) = \exp\left(V_i^T(t)\right)$$

Then the system of ODEs writes

$\forall (i,t) \in \mathcal{I} \times [0,T]$,

$$\frac{d}{dt} w_i^T(t) + h(i) w_i^T(t) + \sum_{j \in \mathcal{V}(i)} e^{-1-b_{ij}} w_j^T(t) = 0$$

with terminal condition $w_i^T(T) = e^{g(i)}, \quad \forall i \in \mathcal{I}$.

This is a system of linear ODEs!

Solution to the ODEs

Solution to the ODEs

Proposition

Let $B = (B_{ij})_{(i,j)\in\mathcal{I}^2}$ be the matrix defined by

$$B_{ij} = \begin{cases} e^{-1-b_{ij}}, & \text{if } j \in \mathcal{V}(i), \\ h(i), & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathfrak{g}$ be the column vector $(e^{g(1)}, \ldots, e^{g(N)})'$.

Then, $w^T : t \in [0, T] \mapsto w^T(t) = e^{B(T-t)}\mathfrak{g}$ is the unique solution to the above system of ODEs

Solution to the ODEs

Proposition

Let $B = (B_{ij})_{(i,j)\in\mathcal{I}^2}$ *be the matrix defined by*

$$B_{ij} = \begin{cases} e^{-1-b_{ij}}, & \text{if } j \in \mathcal{V}(i), \\ h(i), & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathfrak{g}$ *be the column vector* $(e^{g(1)}, \ldots, e^{g(N)})'$.

Then, $w^T : t \in [0, T] \mapsto w^T(t) = e^{B(T-t)}\mathfrak{g}$ *is the unique solution to the above system of ODEs*

Remark: $w^T(t) > 0$ (as a vector) is a consequence of the positiveness of

$$e^{\sup_i |h(i)|(T-t)} w^T(t) = e^{(B+\sup_i |h(i)| I_N)(T-t)}\mathfrak{g} > 0$$

Value function and optimal controls

Value function and optimal controls

Theorem

We have:

- $\forall i \in \mathcal{I}, \forall t \in [0, T], u_i^T(t) = \log(w_i^T(t))$.
- *The optimal controls are given in feedback form by:*

$$\forall i \in \mathcal{I}, \forall j \in \mathcal{V}(i), \forall t \in [0, T], \quad \lambda_t^*(i, j) = e^{-1-b_{ij}} \frac{w_j^T(t)}{w_i^T(t)}.$$

Value function and optimal controls

Theorem

We have:

- $\forall i \in \mathcal{I}, \forall t \in [0, T], u_i^T(t) = \log(w_i^T(t))$.
- *The optimal controls are given in feedback form by:*

$$\forall i \in \mathcal{I}, \forall j \in \mathcal{V}(i), \forall t \in [0, T], \quad \lambda_t^*(i,j) = e^{-1-b_{ij}} \frac{w_j^T(t)}{w_i^T(t)}.$$

A question remains: what can we say about the asymptotic regime?

Value function and optimal controls

Theorem

We have:

- $\forall i \in \mathcal{I}, \forall t \in [0, T], u_i^T(t) = \log(w_i^T(t))$.
- *The optimal controls are given in feedback form by:*

$$\forall i \in \mathcal{I}, \forall j \in \mathcal{V}(i), \forall t \in [0, T], \quad \lambda_t^*(i,j) = e^{-1-b_{ij}} \frac{w_j^T(t)}{w_i^T(t)}.$$

A question remains: what can we say about the asymptotic regime?

We can guess that the ergodic constant γ and the vector ξ are linked to spectral properties of B: a matrix with nonnegative off-diagonal entries.

Classical results on nonnegative matrices

Some definitions

Some definitions

Definition

Given two matrices $A, B \in M_{n,p}(\mathbb{C})$, we say that

- $A \leq B$ if the entries of $B - A$ are all real and nonnegative.
- $A < B$ if the entries of $B - A$ are all real and positive.

We say that A is nonnegative (resp. positive) if $A \geq 0$ (resp. $A > 0$).

Some definitions

Definition

Given two matrices $A, B \in M_{n,p}(\mathbb{C})$, we say that

- $A \leq B$ if the entries of $B - A$ are all real and nonnegative.
- $A < B$ if the entries of $B - A$ are all real and positive.

We say that A is nonnegative (resp. positive) if $A \geq 0$ (resp. $A > 0$).

For $A = (a_{ij})_{ij} \in M_{n,p}(\mathbb{C})$, we define $|A| = (|a_{ij}|)_{ij}$

Some definitions

Definition

Given two matrices $A, B \in M_{n,p}(\mathbb{C})$, we say that

- $A \leq B$ if the entries of $B - A$ are all real and nonnegative.
- $A < B$ if the entries of $B - A$ are all real and positive.

We say that A is nonnegative (resp. positive) if $A \geq 0$ (resp. $A > 0$).

For $A = (a_{ij})_{ij} \in M_{n,p}(\mathbb{C})$, we define $|A| = (|a_{ij}|)_{ij}$

Remark: The definitions apply to column vectors ($p = 1$).

Some definitions

Some definitions

Definition

Given a matrix $A \in M_n(\mathbb{C})$ we define

- $\mathrm{Sp}(A)$ the set of its eigenvalues.
- $\mathrm{Sp}_{\mathbb{R}}(A) = \mathrm{Sp}(A) \cap \mathbb{R}$ the set of its real eigenvalues.
- $\rho(A) = \sup\{|z| | z \in \mathrm{Sp}(A)\}$ the spectral radius of A.

Spectral radius and convergence of powers

Spectral radius and convergence of powers

A first classical result about spectral radius is the following:

Spectral radius and convergence of powers

A first classical result about spectral radius is the following:

Proposition

Let $A \in M_n(\mathbb{C})$.

$$\lim_{m \to +\infty} A^m = 0 \iff \rho(A) < 1$$

Spectral radius and convergence of powers

A first classical result about spectral radius is the following:

Proposition

Let $A \in M_n(\mathbb{C})$.

$$\lim_{m \to +\infty} A^m = 0 \iff \rho(A) < 1$$

Proof.

$\Rightarrow$ is trivial using a Jordan decomposition and looking at diagonal terms.

$\Leftarrow$ Each Jordan block of A writes $\tilde{A} = \lambda I + J$ where J is nilpotent of index p and $|\lambda| < 1$.

Spectral radius and convergence of powers

A first classical result about spectral radius is the following:

Proposition

Let $A \in M_n(\mathbb{C})$.

$$\lim_{m \to +\infty} A^m = 0 \iff \rho(A) < 1$$

Proof.

$\Rightarrow$ is trivial using a Jordan decomposition and looking at diagonal terms.

$\Leftarrow$ Each Jordan block of A writes $\tilde{A} = \lambda I + J$ where J is nilpotent of index p and $|\lambda| < 1$.
We have therefore for $m \geq p$:

$$\tilde{A}^m = \sum_{k=0}^{p-1} C_m^k \lambda^{m-k} J^k \to_{m \to +\infty} 0$$

□

Spectral radius: Gelfand's formula

Spectral radius: Gelfand's formula

Proposition (Gelfand's formula)

Let $A \in M_n(\mathbb{C})$.

$$\rho(A) = \lim_{m \to +\infty} \|A^m\|^{1/m}$$

for any norm on $M_n(\mathbb{C})$.

Spectral radius: Gelfand's formula

Proposition (Gelfand's formula)

Let $A \in M_n(\mathbb{C})$.

$$\rho(A) = \lim_{m \to +\infty} \|A^m\|^{1/m}$$

for any norm on $M_n(\mathbb{C})$.

Proof.

Because of the equivalence of norms, we easily see that the result needs to be proved for one norm only.

Spectral radius: Gelfand's formula

Proposition (Gelfand's formula)

Let $A \in M_n(\mathbb{C})$.

$$\rho(A) = \lim_{m \to +\infty} \|A^m\|^{1/m}$$

for any norm on $M_n(\mathbb{C})$.

Proof.

Because of the equivalence of norms, we easily see that the result needs to be proved for one norm only.

We choose a matrix norm induced by a norm on $\mathbb{R}^n$.

Spectral radius: Gelfand's formula

Proposition (Gelfand's formula)

Let $A \in M_n(\mathbb{C})$.

$$\rho(A) = \lim_{m \to +\infty} \|A^m\|^{1/m}$$

for any norm on $M_n(\mathbb{C})$.

Proof.

Because of the equivalence of norms, we easily see that the result needs to be proved for one norm only.
We choose a matrix norm induced by a norm on $\mathbb{R}^n$.

If x is an eigenvector of A for the eigenvalue λ with $|\lambda| = \rho(A)$, then

$$\rho(A)\|x\| = \|\lambda x\| = \|Ax\| \leq \|A\|\|x\|$$

Spectral radius: Gelfand's formula

Proposition (Gelfand's formula)

Let $A \in M_n(\mathbb{C})$.

$$\rho(A) = \lim_{m \to +\infty} \|A^m\|^{1/m}$$

for any norm on $M_n(\mathbb{C})$.

Proof.

Because of the equivalence of norms, we easily see that the result needs to be proved for one norm only.
We choose a matrix norm induced by a norm on $\mathbb{R}^n$.

If x is an eigenvector of A for the eigenvalue λ with $|\lambda| = \rho(A)$, then

$$\rho(A)\|x\| = \|\lambda x\| = \|Ax\| \leq \|A\|\|x\|$$

So $\rho(A) \leq \|A\|$ and $\rho(A) = \rho(A^m)^{1/m} \leq \|A^m\|^{1/m}$.

Spectral radius: Gelfand's formula

Spectral radius: Gelfand's formula

Proof.

Now, for any $\epsilon > 0$, $\rho\left(\frac{A}{\rho(A)+\epsilon}\right) < 1$. Therefore, there exists $m_\epsilon \in \mathbb{N}$ such that $\forall m \geq m_\epsilon$:

$$\left\|\left(\frac{A}{\rho(A)+\epsilon}\right)^m\right\| \leq 1$$

i.e.

$$\|A^m\|^{1/m} \leq \rho(A) + \epsilon.$$

Spectral radius: Gelfand's formula

Proof.

Now, for any $\epsilon > 0$, $\rho\left(\frac{A}{\rho(A)+\epsilon}\right) < 1$. Therefore, there exists $m_\epsilon \in \mathbb{N}$ such that $\forall m \geq m_\epsilon$:

$$\left\| \left(\frac{A}{\rho(A) + \epsilon} \right)^m \right\| \leq 1$$

i.e.

$$\|A^m\|^{1/m} \leq \rho(A) + \epsilon.$$

We conclude that

$$\lim_{m \to +\infty} \|A^m\|^{1/m} = \rho(A)$$

□

Spectral radius: comparison for nonnegative matrices

Spectral radius: comparison for nonnegative matrices

Proposition

Let $A, B \in M_n(\mathbb{R})$ *and assume* $0 \leq A \leq B$.

Then,

$$\rho(A) \leq \rho(B)$$

Spectral radius: comparison for nonnegative matrices

Proposition

Let $A, B \in M_n(\mathbb{R})$ *and assume* $0 \leq A \leq B$.

Then,

$$\rho(A) \leq \rho(B)$$

Proof.

$$0 \leq A \leq B \Rightarrow 0 \leq A^m \leq B^m \to \|A^m\| \leq \|B^m\|$$

where the norm on matrices is the 2-norm (Frobenius norm).

Spectral radius: comparison for nonnegative matrices

Proposition

Let $A, B \in M_n(\mathbb{R})$ *and assume* $0 \leq A \leq B$.

Then,

$$\rho(A) \leq \rho(B)$$

Proof.

$$0 \leq A \leq B \Rightarrow 0 \leq A^m \leq B^m \rightarrow \|A^m\| \leq \|B^m\|$$

where the norm on matrices is the 2-norm (Frobenius norm).

Using Gelfand's formula, we obtain $\rho(A) \leq \rho(B)$. □

Positive matrices: a first lemma

Positive matrices: a first lemma

We now focus on the case of positive matrices. We have a first (important) lemma:

Positive matrices: a first lemma

We now focus on the case of positive matrices. We have a first (important) lemma:

Lemma

Let $A \in M_n(\mathbb{R})$ be a positive matrix.
Let $x, y \in \mathbb{R}^n$.

$$\begin{aligned} x \leq y \text{ and } x \neq y \quad &\Longrightarrow \quad Ax < Ay \\ &\Longrightarrow \quad \exists \epsilon > 0, (1+\epsilon)Ax < Ay \end{aligned}$$

Positive matrices: a first lemma

We now focus on the case of positive matrices. We have a first (important) lemma:

Lemma

Let $A \in M_n(\mathbb{R})$ be a positive matrix.
Let $x, y \in \mathbb{R}^n$.

$$\begin{aligned} x \leq y \text{ and } x \neq y \quad &\implies \quad Ax < Ay \\ &\implies \quad \exists \epsilon > 0, (1+\epsilon)Ax < Ay \end{aligned}$$

Proof.

For all $i \in \mathcal{I}$,

$$(A(y-x))_i = \sum_{j=1}^{n} A_{ij}(y_j - x_j) \geq \underbrace{\min_k A_{ik}}_{>0} \underbrace{\sum_{j=1}^{n}(y_j - x_j)}_{>0} > 0$$

Positive matrices: a first lemma

We now focus on the case of positive matrices. We have a first (important) lemma:

Lemma

Let $A \in M_n(\mathbb{R})$ be a positive matrix.
Let $x, y \in \mathbb{R}^n$.

$$\begin{aligned} x \leq y \text{ and } x \neq y \quad &\Longrightarrow \quad Ax < Ay \\ &\Longrightarrow \quad \exists \epsilon > 0, (1+\epsilon)Ax < Ay \end{aligned}$$

Proof.

For all $i \in \mathcal{I}$,

$$(A(y-x))_i = \sum_{j=1}^{n} A_{ij}(y_j - x_j) \geq \underbrace{\min_k A_{ik}}_{>0} \underbrace{\sum_{j=1}^{n}(y_j - x_j)}_{>0} > 0$$

So $Ax < Ay$ and there exists $\epsilon > 0$, such that $(1+\epsilon)Ax < Ay$. □

Positive matrices: Perron's theorem

Positive matrices: Perron's theorem

We are now ready to state a fundamental theorem for positive matrices:

Positive matrices: Perron's theorem

We are now ready to state a fundamental theorem for positive matrices:

Theorem (Perron's theorem)

Let $A \in M_n(\mathbb{R})$ be a positive matrix. We have the following:

- $\rho(A) > 0$.
- $\rho(A)$ *is an eigenvalue of* A.
- *the associated eigenspace is of dimension* 1 *and spanned by a positive vector.*
- *the algebraic multiplicity of* $\rho(A)$ *is* 1.

Positive matrices: Perron's theorem

Positive matrices: Perron's theorem

Proof.

$\rho(A) > 0$ as $\mathrm{Tr}(A) > 0$.

Positive matrices: Perron's theorem

Proof.

$\rho(A) > 0$ as $\mathrm{Tr}(A) > 0$.

Let (λ, x) be an eigenpair with $|\lambda| = \rho(A)$.

Positive matrices: Perron's theorem

Proof.

$\rho(A) > 0$ as $\mathrm{Tr}(A) > 0$.

Let (λ, x) be an eigenpair with $|\lambda| = \rho(A)$.

$$Ax = \lambda x \implies \rho(A)|x| = |Ax| \leq A|x|$$

Positive matrices: Perron's theorem

Proof.

$\rho(A) > 0$ as $\mathrm{Tr}(A) > 0$.

Let (λ, x) be an eigenpair with $|\lambda| = \rho(A)$.

$$Ax = \lambda x \implies \rho(A)|x| = |Ax| \leq A|x|$$

If $\rho(A)|x| \neq A|x|$, there exists $\epsilon > 0$ such that

$$(1+\epsilon)\rho(A)A|x| < A^2|x|$$

Positive matrices: Perron's theorem

Proof.

$\rho(A) > 0$ as $\mathrm{Tr}(A) > 0$.

Let (λ, x) be an eigenpair with $|\lambda| = \rho(A)$.

$$Ax = \lambda x \implies \rho(A)|x| = |Ax| \leq A|x|$$

If $\rho(A)|x| \neq A|x|$, there exists $\epsilon > 0$ such that

$$(1+\epsilon)\rho(A)A|x| < A^2|x|$$

So $(1+\epsilon)\rho(A)^2|x| < A^2|x|$ and we can iterate:

$$(1+\epsilon)^2\rho(A)^3|x| = (1+\epsilon)^2\rho(A)^2\rho(A)|x| \leq (1+\epsilon)^2\rho(A)^2A|x| < A^3|x|$$

$$\cdots$$

$$\forall m \geq 2, \quad (1+\epsilon)^{m-1}\rho(A)^m|x| < A^m|x|$$

Positive matrices: Perron's theorem

Positive matrices: Perron's theorem

Proof.

We deduce that for the matrix norm induced by the sup-norm on $\mathbb{R}^n$:

$$\forall m \geq 2, \quad \|A^m\| \geq (1+\epsilon)^{m-1}\rho(A)^m$$

Positive matrices: Perron's theorem

Proof.

We deduce that for the matrix norm induced by the sup-norm on $\mathbb{R}^n$:

$$\forall m \geq 2, \quad \|A^m\| \geq (1+\epsilon)^{m-1}\rho(A)^m$$

Using Gelfand's formula we obtain $\rho(A) \geq (1+\epsilon)\rho(A)$... a contradiction.

Positive matrices: Perron's theorem

Proof.

We deduce that for the matrix norm induced by the sup-norm on $\mathbb{R}^n$:

$$\forall m \geq 2, \quad \|A^m\| \geq (1+\epsilon)^{m-1}\rho(A)^m$$

Using Gelfand's formula we obtain $\rho(A) \geq (1+\epsilon)\rho(A)$... a contradiction. We conclude

$$\rho(A)|x| = A|x|$$

and

$$|x| \geq 0 \implies \rho(A)|x| = A|x| > 0 \implies |x| > 0.$$

Positive matrices: Perron's theorem

Proof.

We deduce that for the matrix norm induced by the sup-norm on $\mathbb{R}^n$:

$$\forall m \geq 2, \quad \|A^m\| \geq (1+\epsilon)^{m-1}\rho(A)^m$$

Using Gelfand's formula we obtain $\rho(A) \geq (1+\epsilon)\rho(A)$... a contradiction. We conclude

$$\rho(A)|x| = A|x|$$

and

$$|x| \geq 0 \implies \rho(A)|x| = A|x| > 0 \implies |x| > 0.$$

Now, if $\tilde{x}$ is another eigenvector for the eigenvalue $\rho(A)$, we have, as before, that $|\tilde{x}|$ is also an eigenvector for the eigenvalue $\rho(A)$, and

$$\rho(A)|\tilde{x}| = |A\tilde{x}| \leq A|\tilde{x}| = \rho(A)|\tilde{x}|$$

Positive matrices: Perron's theorem

Proof.

We deduce that for the matrix norm induced by the sup-norm on $\mathbb{R}^n$:

$$\forall m \geq 2, \quad \|A^m\| \geq (1+\epsilon)^{m-1}\rho(A)^m$$

Using Gelfand's formula we obtain $\rho(A) \geq (1+\epsilon)\rho(A)$... a contradiction. We conclude

$$\rho(A)|x| = A|x|$$

and

$$|x| \geq 0 \implies \rho(A)|x| = A|x| > 0 \implies |x| > 0.$$

Now, if $\tilde{x}$ is another eigenvector for the eigenvalue $\rho(A)$, we have, as before, that $|\tilde{x}|$ is also an eigenvector for the eigenvalue $\rho(A)$, and

$$\rho(A)|\tilde{x}| = |A\tilde{x}| \leq A|\tilde{x}| = \rho(A)|\tilde{x}|$$

So we have an equality case in the triangular inequality $|A\tilde{x}| \leq A|\tilde{x}|$.

Positive matrices: Perron's theorem

Positive matrices: Perron's theorem

Proof.

The first coordinate gives that $\arg(A_{1j}\tilde{x}_j)$ is independent of j. As $A > 0$, we have $\tilde{x} = e^{i\theta}|\tilde{x}|$.

Positive matrices: Perron's theorem

Proof.

The first coordinate gives that $\arg(A_{1j}\tilde{x}_j)$ is independent of j. As $A > 0$, we have $\tilde{x} = e^{i\theta}|\tilde{x}|$.

Now, let us consider $c = \min_{|\tilde{x}_i| \neq 0} |x_i|/|\tilde{x}_i|$.

Positive matrices: Perron's theorem

Proof.

The first coordinate gives that $\arg(A_{1j}\tilde{x}_j)$ is independent of j. As $A > 0$, we have $\tilde{x} = e^{i\theta}|\tilde{x}|$.

Now, let us consider $c = \min_{|\tilde{x}_i|\neq 0} |x_i|/|\tilde{x}_i|$.
If $|x| \neq c|\tilde{x}|$, then

$$|x| \geq c|\tilde{x}| \implies \rho(A)|x| = A|x| > cA|\tilde{x}| = c\rho(A)|\tilde{x}| \implies |x| > c|\tilde{x}|$$

which contradicts the definition of c.

Positive matrices: Perron's theorem

Proof.

The first coordinate gives that $\arg(A_{1j}\tilde{x}_j)$ is independent of j. As $A > 0$, we have $\tilde{x} = e^{i\theta}|\tilde{x}|$.

Now, let us consider $c = \min_{|\tilde{x}_i| \neq 0} |x_i|/|\tilde{x}_i|$.
If $|x| \neq c|\tilde{x}|$, then

$$|x| \geq c|\tilde{x}| \implies \rho(A)|x| = A|x| > cA|\tilde{x}| = c\rho(A)|\tilde{x}| \implies |x| > c|\tilde{x}|$$

which contradicts the definition of c.

We conclude that $|x| = c|\tilde{x}| = ce^{-i\theta}\tilde{x}$, i.e. the eigenspace associated with $\rho(A)$ is of dimension 1.

Positive matrices: Perron's theorem

Positive matrices: Perron's theorem

Proof.

Applying the above reasoning to both A and A', we exhibit two positive vectors u and v such that

$$Au = \rho(A)u \quad \text{and} \quad A'v = \rho(A)v.$$

Positive matrices: Perron's theorem

Proof.

Applying the above reasoning to both A and A', we exhibit two positive vectors u and v such that

$$Au = \rho(A)u \quad \text{and} \quad A'v = \rho(A)v.$$

$u'v > 0$ so $\mathbb{R}^n = \mathrm{span}(u) \oplus \mathrm{span}(v)^\perp$.

Positive matrices: Perron's theorem

Proof.

Applying the above reasoning to both A and A', we exhibit two positive vectors u and v such that

$$Au = \rho(A)u \quad \text{and} \quad A'v = \rho(A)v.$$

$u'v > 0$ so $\mathbb{R}^n = \text{span}(u) \oplus \text{span}(v)^\perp$. Since $\text{span}(v)^\perp$ is stable by A, there exists $P \in GL_n(\mathbb{R})$ such that

$$PAP^{-1} = \begin{pmatrix} \rho(A) & 0 \\ 0 & \tilde{A} \end{pmatrix}$$

Positive matrices: Perron's theorem

Proof.

Applying the above reasoning to both A and A', we exhibit two positive vectors u and v such that

$$Au = \rho(A)u \quad \text{and} \quad A'v = \rho(A)v.$$

$u'v > 0$ so $\mathbb{R}^n = \text{span}(u) \oplus \text{span}(v)^\perp$. Since $\text{span}(v)^\perp$ is stable by A, there exists $P \in GL_n(\mathbb{R})$ such that

$$PAP^{-1} = \begin{pmatrix} \rho(A) & 0 \\ 0 & \tilde{A} \end{pmatrix}$$

As the eigenspace of A associated with $\rho(A)$ is of dimension 1, $\rho(A)$ cannot be an eigenvalue $\tilde{A}$.

Positive matrices: Perron's theorem

Proof.

Applying the above reasoning to both A and A', we exhibit two positive vectors u and v such that

$$Au = \rho(A)u \quad \text{and} \quad A'v = \rho(A)v.$$

$u'v > 0$ so $\mathbb{R}^n = \mathrm{span}(u) \oplus \mathrm{span}(v)^\perp$. Since $\mathrm{span}(v)^\perp$ is stable by A, there exists $P \in GL_n(\mathbb{R})$ such that

$$PAP^{-1} = \begin{pmatrix} \rho(A) & 0 \\ 0 & \tilde{A} \end{pmatrix}$$

As the eigenspace of A associated with $\rho(A)$ is of dimension 1, $\rho(A)$ cannot be an eigenvalue $\tilde{A}$.
We conclude that $\rho(A)$ has algebraic multiplicity 1. □

A first extension to nonnegative matrices

A first extension to nonnegative matrices

A natural question is “what can be generalized to nonnegative matrices?”.

A first extension to nonnegative matrices

A natural question is "what can be generalized to nonnegative matrices?".

A first result is the following:

Proposition

Let $A \in M_n(\mathbb{R})$ be a nonnegative matrix.

Then $\rho(A)$ is an eigenvalue of A and there exists a nonnegative eigenvector associated with $\rho(A)$.

A first extension to nonnegative matrices

A first extension to nonnegative matrices

Proof.

We define $A_p = A + \frac{1}{p} J$ where J is a matrix with all entries equal to 1.

A first extension to nonnegative matrices

Proof.

We define $A_p = A + \frac{1}{p}J$ where J is a matrix with all entries equal to 1. By Perron's theorem, there exists for each $p \geq 1$, a positive vector x_p such that

$$A_p x_p = \rho(A_p) x_p \quad \|x_p\| = 1$$

A first extension to nonnegative matrices

Proof.

We define $A_p = A + \frac{1}{p}J$ where J is a matrix with all entries equal to 1. By Perron's theorem, there exists for each $p \geq 1$, a positive vector x_p such that

$$A_p x_p = \rho(A_p) x_p \quad \|x_p\| = 1$$

We can extract a subsequence $x_{p'} \to x$ with $x \geq 0$ and $\|x\| = 1$.

A first extension to nonnegative matrices

Proof.

We define $A_p = A + \frac{1}{p}J$ where J is a matrix with all entries equal to 1. By Perron's theorem, there exists for each $p \geq 1$, a positive vector x_p such that

$$A_p x_p = \rho(A_p) x_p \quad \|x_p\| = 1$$

We can extract a subsequence $x_{p'} \to x$ with $x \geq 0$ and $\|x\| = 1$.

Because $A \leq A_p \leq A_q$ for $p \geq q$, the sequence $(\rho(A_{p'}))_{p'}$ is nonincreasing and converges towards $\rho \geq \rho(A)$.

A first extension to nonnegative matrices

Proof.

We define $A_p = A + \frac{1}{p}J$ where J is a matrix with all entries equal to 1. By Perron's theorem, there exists for each $p \geq 1$, a positive vector x_p such that

$$A_p x_p = \rho(A_p) x_p \quad \|x_p\| = 1$$

We can extract a subsequence $x_{p'} \to x$ with $x \geq 0$ and $\|x\| = 1$.

Because $A \leq A_p \leq A_q$ for $p \geq q$, the sequence $(\rho(A_{p'}))_{p'}$ is nonincreasing and converges towards $\rho \geq \rho(A)$.

We obtain

$$Ax = \rho x \quad \|x\| = 1 \quad x \geq 0$$

A first extension to nonnegative matrices

Proof.

We define $A_p = A + \frac{1}{p}J$ where J is a matrix with all entries equal to 1. By Perron's theorem, there exists for each $p \geq 1$, a positive vector x_p such that

$$A_p x_p = \rho(A_p) x_p \quad \|x_p\| = 1$$

We can extract a subsequence $x_{p'} \to x$ with $x \geq 0$ and $\|x\| = 1$.

Because $A \leq A_p \leq A_q$ for $p \geq q$, the sequence $(\rho(A_{p'}))_{p'}$ is nonincreasing and converges towards $\rho \geq \rho(A)$.

We obtain

$$Ax = \rho x \quad \|x\| = 1 \quad x \geq 0$$

As $\rho \geq \rho(A)$ is an eigenvalue, we have $\rho = \rho(A)$. □

A song of matrices and graphs

In order to generalize other results, we need an additional assumption: irreducibility.

A song of matrices and graphs

In order to generalize other results, we need an additional assumption: irreducibility.

Let us start with a few definitions:

A song of matrices and graphs

In order to generalize other results, we need an additional assumption: irreducibility.

Let us start with a few definitions:

Definition

For $A \in M_n(\mathbb{C})$ we denote by $M(A)$ the matrix with entries $(1_{a_{ij} \neq 0})_{ij}$.

A song of matrices and graphs

In order to generalize other results, we need an additional assumption: irreducibility.

Let us start with a few definitions:

Definition

For $A \in M_n(\mathbb{C})$ we denote by $M(A)$ the matrix with entries $(1_{a_{ij} \neq 0})_{ij}$.

Definition

For $A \in M_n(\mathbb{C})$ we define $\Gamma(A)$ the directed graph with adjacency matrix $M(A)$

A song of matrices and graphs

In order to generalize other results, we need an additional assumption: irreducibility.

Let us start with a few definitions:

Definition

For $A \in M_n(\mathbb{C})$ we denote by $M(A)$ the matrix with entries $(1_{a_{ij} \neq 0})_{ij}$.

Definition

For $A \in M_n(\mathbb{C})$ we define $\Gamma(A)$ the directed graph with adjacency matrix $M(A)$

We shall relate properties of A with properties of $\Gamma(A)$.

A song of matrices and graphs

A song of matrices and graphs

Lemma

For $A \in M_n(\mathbb{C})$, $m \in \mathbb{N}$, and $1 \leq i, j \leq n$, the three following statements are equivalent:

- $(|A|^m)_{ij} > 0$
- $(M(A)^m)_{ij} > 0$
- *there exists a path a length m from i to j in the graph $\Gamma(A)$.*

A song of matrices and graphs

Lemma

For $A \in M_n(\mathbb{C})$, $m \in \mathbb{N}$, and $1 \leq i, j \leq n$, the three following statements are equivalent:

- $(|A|^m)_{ij} > 0$
- $(M(A)^m)_{ij} > 0$
- *there exists a path a length m from i to j in the graph $\Gamma(A)$.*

Proof.

$$(|A|^m)_{ij} = \sum_{k_1=i,k_2,\ldots,k_{m-1},k_m=j} |a_{k_1k_2}| \cdots |a_{k_{m-1}k_m}|$$

A song of matrices and graphs

Lemma

For $A \in M_n(\mathbb{C})$, $m \in \mathbb{N}$, *and* $1 \leq i, j \leq n$, *the three following statements are equivalent:*

- $(|A|^m)_{ij} > 0$
- $(M(A)^m)_{ij} > 0$
- *there exists a path a length* m *from* i *to* j *in the graph* $\Gamma(A)$.

Proof.

$$(|A|^m)_{ij} = \sum_{k_1=i,k_2,\ldots,k_{m-1},k_m=j} |a_{k_1k_2}| \cdots |a_{k_{m-1}k_m}|$$

So $(|A|^m)_{ij} > 0$ if and only if there exist $k_1 = i, k_2, \ldots, k_{m-1}, k_m = j$ such that $|a_{k_1k_2}|, \ldots, |a_{k_{m-1}k_m}| \neq 0$,

A song of matrices and graphs

Lemma

For $A \in M_n(\mathbb{C})$, $m \in \mathbb{N}$, and $1 \leq i, j \leq n$, the three following statements are equivalent:

- $(|A|^m)_{ij} > 0$
- $(M(A)^m)_{ij} > 0$
- *there exists a path a length m from i to j in the graph $\Gamma(A)$.*

Proof.

$$(|A|^m)_{ij} = \sum_{k_1=i, k_2, \ldots, k_{m-1}, k_m=j} |a_{k_1 k_2}| \cdots |a_{k_{m-1} k_m}|$$

So $(|A|^m)_{ij} > 0$ if and only if there exist $k_1 = i, k_2, \ldots, k_{m-1}, k_m = j$ such that $|a_{k_1 k_2}|, \ldots, |a_{k_{m-1} k_m}| \neq 0$, i.e. if and only if there exists a path a length m from i to j in the graph $\Gamma(A)$.

A song of matrices and graphs

Lemma

For $A \in M_n(\mathbb{C})$, $m \in \mathbb{N}$, and $1 \le i, j \le n$, the three following statements are equivalent:

- $(|A|^m)_{ij} > 0$
- $(M(A)^m)_{ij} > 0$
- *there exists a path a length m from i to j in the graph $\Gamma(A)$.*

Proof.

$$(|A|^m)_{ij} = \sum_{k_1=i, k_2, \dots, k_{m-1}, k_m=j} |a_{k_1 k_2}| \cdots |a_{k_{m-1} k_m}|$$

So $(|A|^m)_{ij} > 0$ if and only if there exist $k_1 = i, k_2, \dots, k_{m-1}, k_m = j$ such that $|a_{k_1 k_2}|, \dots, |a_{k_{m-1} k_m}| \neq 0$, i.e. if and only if there exists a path a length m from i to j in the graph $\Gamma(A)$.

To complete the proof, simply notice that $\Gamma(A) = \Gamma(M(A))$. □

A song of matrices and graphs

A song of matrices and graphs

Proposition

For $A \in M_n(\mathbb{C})$ the three following statements are equivalent:

- $(I_n + |A|)^{n-1} > 0$
- $(I_n + M(A))^{n-1} > 0$
- *The graph $\Gamma(A)$ is connected.*

A song of matrices and graphs

Proposition

For $A \in M_n(\mathbb{C})$ the three following statements are equivalent:

- $(I_n + |A|)^{n-1} > 0$
- $(I_n + M(A))^{n-1} > 0$
- *The graph $\Gamma(A)$ is connected.*

Proof.

$$(I_n + |A|)^{n-1} = \sum_{m=0}^{n-1} C_{n-1}^{m} |A|^m$$

A song of matrices and graphs

Proposition

For $A \in M_n(\mathbb{C})$ the three following statements are equivalent:

- $(I_n + |A|)^{n-1} > 0$
- $(I_n + M(A))^{n-1} > 0$
- *The graph $\Gamma(A)$ is connected.*

Proof.

$$(I_n + |A|)^{n-1} = \sum_{m=0}^{n-1} C_{n-1}^{m} |A|^m$$

So the diagonal entries of $(I_n + |A|)^{n-1}$ are positive and the off-diagonal are positive if and only if for all $1 \leq i \neq j \leq n$, there exists $m \in \{1, \ldots, n-1\}$ such that $(|A|^m)_{ij} > 0$.

A song of matrices and graphs

Proof.

Using the above lemma, we have $(I_n + |A|)^{n-1} > 0$ if and only if any two distinct nodes of $\Gamma(A)$ are linked by a path of length at most equal to $n - 1$.

A song of matrices and graphs

Proof.

Using the above lemma, we have $(I_n + |A|)^{n-1} > 0$ if and only if any two distinct nodes of $\Gamma(A)$ are linked by a path of length at most equal to $n - 1$.

As the graph has n nodes, $(I_n + |A|)^{n-1} > 0$ is equivalent to $\Gamma(A)$ connected.

A song of matrices and graphs

Proof.

Using the above lemma, we have $(I_n + |A|)^{n-1} > 0$ if and only if any two distinct nodes of $\Gamma(A)$ are linked by a path of length at most equal to $n - 1$.

As the graph has n nodes, $(I_n + |A|)^{n-1} > 0$ is equivalent to $\Gamma(A)$ connected.

To complete the proof, simply notice that $\Gamma(A) = \Gamma(M(A))$. □

A song of matrices and graphs

Proof.

Using the above lemma, we have $(I_n + |A|)^{n-1} > 0$ if and only if any two distinct nodes of $\Gamma(A)$ are linked by a path of length at most equal to $n - 1$.

As the graph has n nodes, $(I_n + |A|)^{n-1} > 0$ is equivalent to $\Gamma(A)$ connected.

To complete the proof, simply notice that $\Gamma(A) = \Gamma(M(A))$. □

The matrices verifying any of the three above assumptions are called **irreducible**.

A song of matrices and graphs

Proof.

Using the above lemma, we have $(I_n + |A|)^{n-1} > 0$ if and only if any two distinct nodes of $\Gamma(A)$ are linked by a path of length at most equal to $n-1$.

As the graph has n nodes, $(I_n + |A|)^{n-1} > 0$ is equivalent to $\Gamma(A)$ connected.

To complete the proof, simply notice that $\Gamma(A) = \Gamma(M(A))$. □

The matrices verifying any of the three above assumptions are called **irreducible**.

Remark: This name comes from another characterization with the impossibility to permute lines/columns to obtain a block-triangular matrix (but we shall not use that in what follows).

Nonnegative and irreducible matrices: Perron-Frobenius theorem

Nonnegative and irreducible matrices: Perron-Frobenius theorem

A fundamental theorem for nonnegative and irreducible matrices is Perron-Frobenius theorem stating that Perron's theorem generalizes to these matrices:

Nonnegative and irreducible matrices: Perron-Frobenius theorem

A fundamental theorem for nonnegative and irreducible matrices is Perron-Frobenius theorem stating that Perron's theorem generalizes to these matrices:

Theorem (Perron-Frobenius theorem)

Let $A \in M_n(\mathbb{R})$ be a nonnegative and irreducible matrix. We have the following:

- $\rho(A) > 0$
- $\rho(A)$ *is an eigenvalue of* A
- *the associated eigenspace is of dimension* 1 *and spanned by a positive vector.*
- *the algebraic multiplicity of* $\rho(A)$ *is* 1*.*

Nonnegative and irreducible matrices: Perron-Frobenius theorem

Nonnegative and irreducible matrices: Perron-Frobenius theorem

Proof.

$\rho(A) = 0 \implies A$ nilpotent $\implies \exists m, A^m = |A|^m = 0.$

Nonnegative and irreducible matrices: Perron-Frobenius theorem

Proof.

$\rho(A) = 0 \implies A$ nilpotent $\implies \exists m, A^m = |A|^m = 0$.
However, because $\Gamma(A)$ is connected, there exist paths of any length in the graph, so $\rho(A) > 0$.

Nonnegative and irreducible matrices: Perron-Frobenius theorem

Proof.

$\rho(A) = 0 \implies A$ nilpotent $\implies \exists m, A^m = |A|^m = 0$.
However, because $\Gamma(A)$ is connected, there exist paths of any length in the graph, so $\rho(A) > 0$.

The second point of the theorem does not require irreducibility (see above).

Nonnegative and irreducible matrices: Perron-Frobenius theorem

Proof.

$\rho(A) = 0 \implies A$ nilpotent $\implies \exists m, A^m = |A|^m = 0$.
However, because $\Gamma(A)$ is connected, there exist paths of any length in the graph, so $\rho(A) > 0$.

The second point of the theorem does not require irreducibility (see above). Let $x \geq 0$ be such that $Ax = \rho(A)x$. Then

$$(I + |A|)^{n-1}x = (I + A)^{n-1}x = (1 + \rho(A))^{n-1}x$$

But

$$\rho((I + |A|)^{n-1}) = \rho(I + |A|)^{n-1} = \rho(I + A)^{n-1} \leq (1 + \rho(A))^{n-1}.$$

Nonnegative and irreducible matrices: Perron-Frobenius theorem

Proof.

$\rho(A) = 0 \implies A$ nilpotent $\implies \exists m, A^m = |A|^m = 0$.
However, because $\Gamma(A)$ is connected, there exist paths of any length in the graph, so $\rho(A) > 0$.

The second point of the theorem does not require irreducibility (see above). Let $x \geq 0$ be such that $Ax = \rho(A)x$. Then

$$(I + |A|)^{n-1}x = (I + A)^{n-1}x = (1 + \rho(A))^{n-1}x$$

But

$$\rho((I + |A|)^{n-1}) = \rho(I + |A|)^{n-1} = \rho(I + A)^{n-1} \leq (1 + \rho(A))^{n-1}.$$

So x is in fact an eigenvalue of $(I + |A|)^{n-1}$ corresponding to its spectral radius. □

Nonnegative and irreducible matrices: Perron-Frobenius theorem

Nonnegative and irreducible matrices: Perron-Frobenius theorem

Proof.

By Perron's theorem, $x > 0$ and the eigenspace of A corresponding to $\rho(A)$ is of dimension 1.

Nonnegative and irreducible matrices: Perron-Frobenius theorem

Proof.

By Perron's theorem, $x > 0$ and the eigenspace of A corresponding to $\rho(A)$ is of dimension 1.

Because A irreducible implies A' irreducible, we can apply the above results to A' and conclude for the fourth point as in the proof of Perron's theorem. $\square$

Nonnegative and irreducible matrices: Perron-Frobenius theorem

Proof.

By Perron's theorem, $x > 0$ and the eigenspace of A corresponding to $\rho(A)$ is of dimension 1.

Because A irreducible implies A' irreducible, we can apply the above results to A' and conclude for the fourth point as in the proof of Perron's theorem. □

Remark: With positive matrices, $\rho(A)$ is the unique eigenvalue with modulus equal to $\rho(A)$. This is not anymore true for nonnegative matrices.

Nonnegative and irreducible matrices: Perron-Frobenius theorem

Proof.

By Perron's theorem, $x > 0$ and the eigenspace of A corresponding to $\rho(A)$ is of dimension 1.

Because A irreducible implies A' irreducible, we can apply the above results to A' and conclude for the fourth point as in the proof of Perron's theorem. □

Remark: With positive matrices, $\rho(A)$ is the unique eigenvalue with modulus equal to $\rho(A)$. This is not anymore true for nonnegative matrices. However we can prove that, if there are several such eigenvalues in the nonnegative and irreducible case, they form a polygon inside the circle of radius $\rho(A)$ in the complex plane.

Entropic costs: spectral characterization of the ergodic constant

Towards asymptotic results

Let us recall that the value function and the optimal controls depend on

$$w^T : t \in [0, T] \mapsto w^T(t) = e^{B(T-t)}\mathfrak{g}$$

where

$$\mathfrak{g} = (e^{g(1)}, \dots, e^{g(N)})'$$

and

$$B_{ij} = \begin{cases} e^{-1-b_{ij}}, & \text{if } j \in \mathcal{V}(i), \\ h(i), & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

Towards asymptotic results

Let us recall that the value function and the optimal controls depend on

$$w^T : t \in [0, T] \mapsto w^T(t) = e^{B(T-t)}\mathfrak{g}$$

where

$$\mathfrak{g} = (e^{g(1)}, \dots, e^{g(N)})'$$

and

$$B_{ij} = \begin{cases} e^{-1-b_{ij}}, & \text{if } j \in \mathcal{V}(i), \\ h(i), & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

We now study the spectrum and deduce the asymptotic behavior of the value function and the optimal controls.

The spectrum of B and asymptotic results

The spectrum of B and asymptotic results

Theorem

$Sp_{\mathbb{R}}(B)$ is a nonempty set and $\gamma = \max Sp_{\mathbb{R}}(B)$ is an algebraically simple eigenvalue whose associated eigenspace is spanned by a positive vector f.

Moreover $\forall \lambda \in Sp(B) \setminus \{\gamma\}, Re(\lambda) < \gamma$.

γ is the ergodic constant associated with our control problem and

$$\exists \alpha \in \mathbb{R}, \forall i \in \mathcal{I}, \forall t \in \mathbb{R}, \quad \lim_{T \to +\infty} u_i^T(t) - \gamma(T - t) = \alpha + \log(f_i).$$

Moreover, the asymptotic behavior of the optimal controls is given by

$$\forall i \in \mathcal{I}, \forall j \in \mathcal{V}(i), \forall t \in \mathbb{R}, \quad \lim_{T \to +\infty} \lambda_t^*(i,j) = e^{-1-b_{ij}} \frac{f_j}{f_i}.$$

Spectrum of B and asymptotic results

Spectrum of B and asymptotic results

Proof.

Let us consider $\sigma = -\min_{i \in \mathcal{I}} h(i)$ and denote by $B(\sigma)$ the nonnegative matrix $B + \sigma I_N$.

Spectrum of B and asymptotic results

Proof.

Let us consider $\sigma = -\min_{i \in \mathcal{I}} h(i)$ and denote by $B(\sigma)$ the nonnegative matrix $B + \sigma I_N$.

$\Gamma(B(\sigma))$ is the connected graph of our problem to which self-loops may have been added: it is connected and therefore $B(\sigma)$ is irreducible.

Spectrum of B and asymptotic results

Proof.

Let us consider $\sigma = -\min_{i \in \mathcal{I}} h(i)$ and denote by $B(\sigma)$ the nonnegative matrix $B + \sigma I_N$.

$\Gamma(B(\sigma))$ is the connected graph of our problem to which self-loops may have been added: it is connected and therefore $B(\sigma)$ is irreducible.

By Perron-Frobenius theorem, $\rho(B(\sigma))$ is an algebraically simple eigenvalue of $B(\sigma)$ and the associated eigenspace is spanned by a positive vector f.

Spectrum of B and asymptotic results

Proof.

Let us consider $\sigma = -\min_{i \in \mathcal{I}} h(i)$ and denote by $B(\sigma)$ the nonnegative matrix $B + \sigma I_N$.

$\Gamma(B(\sigma))$ is the connected graph of our problem to which self-loops may have been added: it is connected and therefore $B(\sigma)$ is irreducible.

By Perron-Frobenius theorem, $\rho(B(\sigma))$ is an algebraically simple eigenvalue of $B(\sigma)$ and the associated eigenspace is spanned by a positive vector f.

Shifting the spectrum by $-\sigma$ we see that $\mathrm{Sp}_{\mathbb{R}}(B)$ is a nonempty set and its maximum γ, equal to $\rho(B(\sigma)) - \sigma$, is an algebraically simple eigenvalue of B whose associated eigenspace is spanned by f.

Spectrum of B and asymptotic results

Proof.

Let us consider $\sigma = -\min_{i \in \mathcal{I}} h(i)$ and denote by $B(\sigma)$ the nonnegative matrix $B + \sigma I_N$.

$\Gamma(B(\sigma))$ is the connected graph of our problem to which self-loops may have been added: it is connected and therefore $B(\sigma)$ is irreducible.

By Perron-Frobenius theorem, $\rho(B(\sigma))$ is an algebraically simple eigenvalue of $B(\sigma)$ and the associated eigenspace is spanned by a positive vector f.

Shifting the spectrum by $-\sigma$ we see that $\mathrm{Sp}_{\mathbb{R}}(B)$ is a nonempty set and its maximum γ, equal to $\rho(B(\sigma)) - \sigma$, is an algebraically simple eigenvalue of B whose associated eigenspace is spanned by f.

Moreover $\forall \lambda \in \mathrm{Sp}(B) \setminus \{\gamma\}, \mathrm{Re}(\lambda) < \gamma$.

Spectrum of B and asymptotic results

Proof.

Now, $\rho(B(\sigma))$ is also an algebraically simple eigenvalue of $B(\sigma)'$ and the associated eigenspace is spanned by a positive vector ϕ.

Spectrum of B and asymptotic results

Proof.

Now, $\rho(B(\sigma))$ is also an algebraically simple eigenvalue of $B(\sigma)'$ and the associated eigenspace is spanned by a positive vector ϕ.

Using a Jordan decomposition of $B(\sigma)$, we see that $\mathfrak{g}$ can be written as $\beta f + \psi$ where $\beta \in \mathbb{R}$ and
$\psi \in \mathrm{Im}(B(\sigma) - \rho(B(\sigma))I_N) = \mathrm{Ker}(B(\sigma)' - \rho(B(\sigma))I_N)^{\perp} = \mathrm{span}(\phi)^{\perp}$.

Spectrum of B and asymptotic results

Proof.

Now, $\rho(B(\sigma))$ is also an algebraically simple eigenvalue of $B(\sigma)'$ and the associated eigenspace is spanned by a positive vector ϕ.

Using a Jordan decomposition of $B(\sigma)$, we see that $\mathfrak{g}$ can be written as $\beta f + \psi$ where $\beta \in \mathbb{R}$ and
$\psi \in \mathrm{Im}(B(\sigma) - \rho(B(\sigma))I_N) = \mathrm{Ker}(B(\sigma)' - \rho(B(\sigma))I_N)^\perp = \mathrm{span}(\phi)^\perp$.

As $\psi = \mathfrak{g} - \beta f \perp \phi$ and all coefficients of $\mathfrak{g}$, f, and ϕ are positive, we must have $\beta > 0$.

Spectrum of B and asymptotic results

Spectrum of B and asymptotic results

Proof.

Now,

$$\begin{aligned} e^{-\gamma(T-t)} w^T(t) &= e^{(B-\gamma I_N)(T-t)} \mathfrak{g} \\ &= e^{(B-\gamma I_N)(T-t)} \beta f + e^{(B-\gamma I_N)(T-t)} \psi \\ &= \beta f + e^{(B-\gamma I_N)(T-t)} \psi \rightarrow_{T\rightarrow+\infty} \beta f. \end{aligned}$$

Spectrum of B and asymptotic results

Proof.

Now,

$$\begin{aligned} e^{-\gamma(T-t)} w^T(t) &= e^{(B-\gamma I_N)(T-t)} \mathfrak{g} \\ &= e^{(B-\gamma I_N)(T-t)} \beta f + e^{(B-\gamma I_N)(T-t)} \psi \\ &= \beta f + e^{(B-\gamma I_N)(T-t)} \psi \to_{T \to +\infty} \beta f. \end{aligned}$$

By taking logarithms, we obtain that

$$\forall i \in \mathcal{I}, \quad \lim_{T \to +\infty} u_i^T(t) - \gamma(T-t) = \log(\beta) + \log(f_i).$$

Spectrum of B and asymptotic results

Proof.

Now,

$$\begin{aligned} e^{-\gamma(T-t)}w^T(t) &= e^{(B-\gamma I_N)(T-t)}\mathfrak{g} \\ &= e^{(B-\gamma I_N)(T-t)}\beta f + e^{(B-\gamma I_N)(T-t)}\psi \\ &= \beta f + e^{(B-\gamma I_N)(T-t)}\psi \to_{T\to+\infty} \beta f. \end{aligned}$$

By taking logarithms, we obtain that

$$\forall i \in \mathcal{I}, \quad \lim_{T\to+\infty} u_i^T(t) - \gamma(T-t) = \log(\beta) + \log(f_i).$$

For optimal controls, we obtain $\forall i \in \mathcal{I}, \forall j \in \mathcal{V}(i), \forall t \in [0,T]$,

$$\begin{aligned} \lambda_t^*(i,j) &= e^{-1-b_{ij}}\frac{w_j^T(t)}{w_i^T(t)} \\ &= e^{-1-b_{ij}}\frac{e^{-\gamma(T-t)}w_j^T(t)}{e^{-\gamma(T-t)}w_i^T(t)} \to_{T\to+\infty} e^{-1-b_{ij}}\frac{f_j}{f_i}. \end{aligned}$$

□

Conclusions about optimal controls on graphs

Conclusions about optimal controls on graphs

What we have seen

Conclusions about optimal controls on graphs

What we have seen

- We have provided, under simple assumptions, a way to characterize optimal controls (with ODEs).

Conclusions about optimal controls on graphs

What we have seen

- We have provided, under simple assumptions, a way to characterize optimal controls (with ODEs).
- We have generalized the results to the case of infinite horizon problems when $r > 0$ (stationary problems).

Conclusions about optimal controls on graphs

What we have seen

- We have provided, under simple assumptions, a way to characterize optimal controls (with ODEs).
- We have generalized the results to the case of infinite horizon problems when $r > 0$ (stationary problems).
- We have obtained a (difficult) result on the asymptotic behavior far from T when $r = 0$.

Conclusions about optimal controls on graphs

What we have seen

- We have provided, under simple assumptions, a way to characterize optimal controls (with ODEs).
- We have generalized the results to the case of infinite horizon problems when $r > 0$ (stationary problems).
- We have obtained a (difficult) result on the asymptotic behavior far from T when $r = 0$.
- We have shown in the case of entropic costs that value functions and optimal controls could be found in closed-form

Conclusions about optimal controls on graphs

What we have seen

- We have provided, under simple assumptions, a way to characterize optimal controls (with ODEs).
- We have generalized the results to the case of infinite horizon problems when $r > 0$ (stationary problems).
- We have obtained a (difficult) result on the asymptotic behavior far from T when $r = 0$.
- We have shown in the case of entropic costs that value functions and optimal controls could be found in closed-form
- We have shown in the case of entropic costs that the ergodic constant is the largest real eigenvalue of a simple matrix and that optimal controls are characterized by the coordinates of an associate eigenvector.

Conclusions about optimal controls on graphs

What we have seen

- We have provided, under simple assumptions, a way to characterize optimal controls (with ODEs).
- We have generalized the results to the case of infinite horizon problems when $r > 0$ (stationary problems).
- We have obtained a (difficult) result on the asymptotic behavior far from T when $r = 0$.
- We have shown in the case of entropic costs that value functions and optimal controls could be found in closed-form
- We have shown in the case of entropic costs that the ergodic constant is the largest real eigenvalue of a simple matrix and that optimal controls are characterized by the coordinates of an associate eigenvector.

We now apply our results to market making and to the Avellaneda-Stoikov equation.

An application to market making

Nature of the problem

Nature of the problem

A problem coming from the financial industry

Nature of the problem

A problem coming from the financial industry

- Not a pricing issue.

Nature of the problem

A problem coming from the financial industry

- Not a pricing issue.
- Not a hedging issue.

Nature of the problem

A problem coming from the financial industry

- Not a pricing issue.
- Not a hedging issue.
- Not a problem of portfolio choice.

Nature of the problem

A problem coming from the financial industry

- Not a pricing issue.
- Not a hedging issue.
- Not a problem of portfolio choice.
- Optimization problem relevant on many markets: market making.

Nature of the problem

A problem coming from the financial industry

- Not a pricing issue.
- Not a hedging issue.
- Not a problem of portfolio choice.
- Optimization problem relevant on many markets: market making.

What is a market maker?

Nature of the problem

A problem coming from the financial industry

- Not a pricing issue.
- Not a hedging issue.
- Not a problem of portfolio choice.
- Optimization problem relevant on many markets: market making.

What is a market maker?

- Liquidity provider: provide bid and ask/offer prices to other market participants

Nature of the problem

A problem coming from the financial industry

- Not a pricing issue.
- Not a hedging issue.
- Not a problem of portfolio choice.
- Optimization problem relevant on many markets: market making.

What is a market maker?

- Liquidity provider: provide bid and ask/offer prices to other market participants
- Today, replaced by algorithms.

Setup of models à la Avellaneda-Stoikov

Setup of models à la Avellaneda-Stoikov

- Reference price process (mid-price) $(S_t)_t$:

$$dS_t = \sigma dW_t.$$

Setup of models à la Avellaneda-Stoikov

- Reference price process (mid-price) $(S_t)_t$:

$$dS_t = \sigma dW_t.$$

- Bid and ask prices of the MM denoted respectively

$$S_t^b = S_t - \delta_t^b \text{ and } S_t^a = S_t + \delta_t^a.$$

Setup of models à la Avellaneda-Stoikov

- Reference price process (mid-price) $(S_t)_t$:

$$dS_t = \sigma dW_t.$$

- Bid and ask prices of the MM denoted respectively

$$S_t^b = S_t - \delta_t^b \text{ and } S_t^a = S_t + \delta_t^a.$$

- Point processes N^b and N^a for the transactions (size Δ). Inventory $(q_t)_t$:

$$dq_t = \Delta dN_t^b - \Delta dN_t^a.$$

Setup of models à la Avellaneda-Stoikov

Setup of models à la Avellaneda-Stoikov

- The intensities of N^b and N^a depend on the distance to the reference price:

$$\lambda_t^b = \Lambda^b(\delta_t^b) 1_{q_{t-} < Q} \text{ and } \lambda_t^a = \Lambda^a(\delta_t^a) 1_{q_{t-} > -Q}.$$

Λ^b, Λ^a decreasing.

Setup of models à la Avellaneda-Stoikov

- The intensities of N^b and N^a depend on the distance to the reference price:

$$\lambda_t^b = \Lambda^b(\delta_t^b)1_{q_{t-}<Q} \text{ and } \lambda_t^a = \Lambda^a(\delta_t^a)1_{q_{t-}>-Q}.$$

Λ^b, Λ^a decreasing.

- Cash process $(X_t)_t$:

$$dX_t = \Delta S_t^a dN_t^a - \Delta S_t^b dN_t^b = -S_t dq_t + \delta_t^a \Delta dN_t^a + \delta_t^b \Delta dN_t^b.$$

Setup of models à la Avellaneda-Stoikov

- The intensities of N^b and N^a depend on the distance to the reference price:

$$\lambda_t^b = \Lambda^b(\delta_t^b) 1_{q_{t-} < Q} \text{ and } \lambda_t^a = \Lambda^a(\delta_t^a) 1_{q_{t-} > -Q}.$$

Λ^b, Λ^a decreasing.

- Cash process $(X_t)_t$:

$$dX_t = \Delta S_t^a dN_t^a - \Delta S_t^b dN_t^b = -S_t dq_t + \delta_t^a \Delta dN_t^a + \delta_t^b \Delta dN_t^b.$$

Three state variables: X (cash), q (inventory), and S (price).

Several objective functions

Naïve: Risk-neutral

$$\sup_{(\delta^a_t)_t,(\delta^b_t)_t \in \mathcal{A}} \mathbb{E}\left[X_T + q_T S_T\right].$$

Several objective functions

Naïve: Risk-neutral

$$\sup_{(\delta_t^a)_t,(\delta_t^b)_t\in\mathcal{A}} \mathbb{E}\left[X_T + q_T S_T\right].$$

The original Avellaneda-Stoikov's model considers a CARA utility function:

CARA objective function (Model A)

$$\sup_{(\delta_t^a)_t,(\delta_t^b)_t\in\mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma(X_T + q_T S_T)\right)\right],$$

where γ is the absolute risk aversion parameter, and $\mathcal{A}$ the set of predictable processes bounded from below.

Several objective functions

Several objective functions

Models à la Cartea, Jaimungal *et al.* with a running penalty for the inventory:

Risk-neutral with running penalty (Model B)

$$\sup_{(\delta_t^a)_t,(\delta_t^b)_t \in \mathcal{A}} \mathbb{E}\left[X_T + q_T S_T - \frac{\gamma}{2}\sigma^2 \int_0^T q_t^2 dt\right],$$

where γ is a kind of absolute risk aversion parameter.

HJB equation (Model A)

HJB equation (Model A)

In what follows, u is a candidate for the value function.

Hamilton-Jacobi-Bellman

$$\text{(HJB)} \qquad 0 = \partial_t u(t,x,q,S) + \frac{1}{2}\sigma^2 \partial^2_{SS} u(t,x,q,S)$$

$$+1_{q<Q} \sup_{\delta^b} \Lambda^b(\delta^b) \left[u(t, x - \Delta S + \Delta\delta^b, q + \Delta, S) - u(t,x,q,S)\right]$$

$$+1_{q>-Q} \sup_{\delta^a} \Lambda^a(\delta^a) \left[u(t, x + \Delta S + \Delta\delta^a, q - \Delta, S) - u(t,x,q,S)\right]$$

with final condition:

$$u(T,x,q,S) = -\exp\left(-\gamma(x + qS)\right)$$

Change of variables (Model A)

Change of variables (Model A)

Ansatz

$$u(t, x, q, S) = -\exp(-\gamma(x + qS + \theta(t, q)))$$

Change of variables (Model A)

Ansatz

$$u(t, x, q, S) = -\exp(-\gamma(x + qS + \theta(t, q)))$$

New equation (Model A)

$$0 = \partial_t \theta(t, q) - \frac{1}{2}\gamma\sigma^2 q^2$$

$$+1_{q<Q} \sup_{\delta^b} \frac{\Lambda^b(\delta^b)}{\gamma} \left(1 - \exp\left(-\gamma\left(\Delta\delta^b + \theta(t, q + \Delta) - \theta(t, q)\right)\right)\right)$$

$$+1_{q>-Q} \sup_{\delta^a} \frac{\Lambda^a(\delta^a)}{\gamma} \left(1 - \exp\left(-\gamma\left(\Delta\delta^a + \theta(t, q - \Delta) - \theta(t, q)\right)\right)\right)$$

with final condition $\theta(T, q) = 0$.

Equation for θ (Model A)

A new transform

$$H^b_\xi(p) = \sup_\delta \frac{\Lambda^b(\delta)}{\xi}\left(1 - \exp\left(-\xi\Delta\left(\delta - p\right)\right)\right)$$

$$H^a_\xi(p) = \sup_\delta \frac{\Lambda^a(\delta)}{\xi}\left(1 - \exp\left(-\xi\Delta\left(\delta - p\right)\right)\right)$$

Equation for θ (Model A)

A new transform

$$H^b_\xi(p) = \sup_\delta \frac{\Lambda^b(\delta)}{\xi}\left(1 - \exp\left(-\xi\Delta\left(\delta - p\right)\right)\right)$$

$$H^a_\xi(p) = \sup_\delta \frac{\Lambda^a(\delta)}{\xi}\left(1 - \exp\left(-\xi\Delta\left(\delta - p\right)\right)\right)$$

New equation (Model A)

$$0 = \partial_t\theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + 1_{q<Q} H^b_\gamma\left(\frac{\theta(t,q) - \theta(t,q+\Delta)}{\Delta}\right)$$
$$+1_{q>-Q} H^a_\gamma\left(\frac{\theta(t,q) - \theta(t,q-\Delta)}{\Delta}\right)$$

with final condition $\theta(T,q) = 0$.

HJB equation (Model B)

HJB equation (Model B)

Hamilton-Jacobi-Bellman

$$\text{(HJB)} \qquad 0 = \partial_t u(t,x,q,S) - \frac{1}{2}\gamma\sigma^2 q^2 + \frac{1}{2}\sigma^2 \partial^2_{SS} u(t,x,q,S)$$

$$+1_{q<Q} \sup_{\delta^b} \Lambda^b(\delta^b)\left[u(t, x - \Delta S + \Delta\delta^b, q + \Delta, S) - u(t,x,q,S)\right]$$

$$+1_{q>-Q} \sup_{\delta^a} \Lambda^a(\delta^a)\left[u(t, x + \Delta S + \Delta\delta^a, q - \Delta, S) - u(t,x,q,S)\right]$$

with final condition:

$$u(T,x,q,S) = x + qS$$

Change of variables (Model B)

Change of variables (Model B)

Ansatz

$$u(T, x, q, S) = x + qS + \theta(t, q)$$

Change of variables (Model B)

Ansatz

$$u(T, x, q, S) = x + qS + \theta(t, q)$$

New equation (Model B)

$$0 = \partial_t \theta(t, q) - \frac{1}{2}\gamma\sigma^2 q^2$$

$$+1_{q<Q} \sup_{\delta^b} \Lambda^b(\delta^b) \left[\Delta\delta^b + \theta(t, q + \Delta) - \theta(t, q)\right]$$

$$+1_{q>-Q} \sup_{\delta^a} \Lambda^a(\delta^a) \left[\Delta\delta^a + \theta(t, q - \Delta) - \theta(t, q)\right]$$

with final condition $\theta(T, q) = 0$.

Equation for θ (Model B)

Equation for θ (Model B)

A new transform

$$H_0^b(p) = \Delta \sup_\delta \Lambda^b(\delta)(\delta - p)$$

$$H_0^a(p) = \Delta \sup_\delta \Lambda^a(\delta)(\delta - p)$$

Equation for θ (Model B)

A new transform

$$H_0^b(p) = \Delta \sup_\delta \Lambda^b(\delta)(\delta - p)$$

$$H_0^a(p) = \Delta \sup_\delta \Lambda^a(\delta)(\delta - p)$$

New equation (Model B)

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + 1_{q<Q} H_0^b\left(\frac{\theta(t,q) - \theta(t,q+\Delta)}{\Delta}\right)$$
$$+1_{q>-Q} H_0^a\left(\frac{\theta(t,q) - \theta(t,q-\Delta)}{\Delta}\right)$$

with final condition $\theta(T,q) = 0$.

A unique family of equations

A unique family of equations

Uniting two objective functions

A unique family of equations

Uniting two objective functions

- Same family of equations for θ in both models.

A unique family of equations

Uniting two objective functions

- Same family of equations for θ in both models.
- A system of $2Q/\Delta + 1$ non-linear ODEs.

A unique family of equations

Uniting two objective functions

- Same family of equations for θ in both models.
- A system of $2Q/\Delta + 1$ non-linear ODEs.
- In both cases: problem in dimension 2 instead of 4.

A unique family of equations

Uniting two objective functions

- Same family of equations for θ in both models.
- A system of $2Q/\Delta + 1$ non-linear ODEs.
- In both cases: problem in dimension 2 instead of 4.

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + 1_{q<Q} H^b_\xi\left(\frac{\theta(t,q) - \theta(t,q+\Delta)}{\Delta}\right)$$
$$+1_{q>-Q} H^a_\xi\left(\frac{\theta(t,q) - \theta(t,q-\Delta)}{\Delta}\right)$$

with final condition $\theta(T,q) = 0$.

A unique family of equations

Uniting two objective functions

- Same family of equations for θ in both models.
- A system of $2Q/\Delta + 1$ non-linear ODEs.
- In both cases: problem in dimension 2 instead of 4.

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + 1_{q<Q} H^b_\xi \left(\frac{\theta(t,q) - \theta(t,q+\Delta)}{\Delta} \right)$$
$$+ 1_{q>-Q} H^a_\xi \left(\frac{\theta(t,q) - \theta(t,q-\Delta)}{\Delta} \right)$$

with final condition $\theta(T,q) = 0$.

Same equations as those studied earlier (written in a slightly different manner)

The intensity functions Λ^b and Λ^a

The intensity functions Λ^b and Λ^a

Assumptions on Λ^b and Λ^a.

1. $\Lambda^{b/a}$ is C^2.
2. ${\Lambda^{b/a}}' < 0$.
3. $\lim_{\delta \to +\infty} \Lambda^{b/a}(\delta) = 0$.
4. The intensity functions $\Lambda^{b/a}$ satisfy:

$$\sup_{\delta} \frac{\Lambda^{b/a}(\delta){\Lambda^{b/a}}''(\delta)}{\left({\Lambda^{b/a}}'(\delta)\right)^2} < 2.$$

The intensity functions Λ^b and Λ^a

Assumptions on Λ^b and Λ^a.

1. $\Lambda^{b/a}$ is C^2.
2. $\Lambda^{b/a'} < 0$.
3. $\lim_{\delta \to +\infty} \Lambda^{b/a}(\delta) = 0$.
4. The intensity functions $\Lambda^{b/a}$ satisfy:

$$\sup_{\delta} \frac{\Lambda^{b/a}(\delta)\Lambda^{b/a''}(\delta)}{\left(\Lambda^{b/a'}(\delta)\right)^2} < 2.$$

Exponential intensity

In Avellaneda and Stoikov ($\Delta = 1$):

$$\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}.$$

The functions H_ξ^b and H_ξ^a

Proposition

- $\forall \xi \geq 0$, $H_\xi^{b/a}$ is a decreasing function of class C^2.
- In the definition of $H_\xi^{b/a}(p)$, the supremum is attained at a unique $\tilde{\delta}_\xi^{b/a*}(p)$ characterized by

$$\tilde{\delta}_\xi^{b/a*}(p) = {\Lambda^{b/a}}^{-1}\left(\xi H_\xi^{b/a}(p) - \frac{{H_\xi^{b/a}}'(p)}{\Delta}\right).$$

- The function $p \mapsto \tilde{\delta}_\xi^{b/a*}(p)$ is increasing.

The functions H_ξ^b and H_ξ^a

Proposition

- $\forall \xi \geq 0$, $H_\xi^{b/a}$ is a decreasing function of class C^2.
- In the definition of $H_\xi^{b/a}(p)$, the supremum is attained at a unique $\tilde{\delta}_\xi^{b/a*}(p)$ characterized by

$$\tilde{\delta}_\xi^{b/a*}(p) = \Lambda^{b/a^{-1}} \left(\xi H_\xi^{b/a}(p) - \frac{H_\xi^{b/a'}(p)}{\Delta} \right).$$

- The function $p \mapsto \tilde{\delta}_\xi^{b/a*}(p)$ is increasing.

Remark: $H_\xi^{b/a}$ decreasing corresponds to increasing Hamiltonian functions in our optimal control theory on graphs.

Existence and uniqueness

Existence and uniqueness

Results for θ

There exists a unique C^1 (in time) solution $t \mapsto (\theta(t,q))_{|q|\leq Q}$ to

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 + 1_{q<Q} H^b_\xi \left(\frac{\theta(t,q) - \theta(t,q+\Delta)}{\Delta} \right)$$

$$+ 1_{q>-Q} H^a_\xi \left(\frac{\theta(t,q) - \theta(t,q-\Delta)}{\Delta} \right)$$

with final condition $\theta(T,q) = 0$.

Solution of the initial problems (verification argument)

Solution of the initial problems (verification argument)

By using a verification argument, the functions u are the value functions associated with the problems of Model A and Model B.

Optimal quotes

The optimal quotes in models A ($\xi = \gamma$) and B ($\xi = 0$) are:

$$\delta_t^{b*} = \tilde{\delta}_\xi^{b*}\left(\frac{\theta(t, q_{t-}) - \theta(t, q_{t-} + \Delta)}{\Delta}\right)$$

$$\delta_t^{a*} = \tilde{\delta}_\xi^{a*}\left(\frac{\theta(t, q_{t-}) - \theta(t, q_{t-} - \Delta)}{\Delta}\right)$$

where

$$\tilde{\delta}_\xi^{b/a*}(p) = {\Lambda^{b/a}}^{-1}\left(\xi H_\xi^{b/a}(p) - \frac{{H_\xi^{b/a}}'(p)}{\Delta}\right).$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The functions $H_\xi^{b/a}$ and $\tilde{\delta}_\xi^{b/a*}$

If $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$, then $H_\xi^{b/a}(p) = \frac{A\Delta}{k} C_\xi \exp(-kp)$, with

$$C_\xi = \begin{cases} \left(1 + \frac{\xi\Delta}{k}\right)^{-\frac{k}{\xi\Delta}-1} & \text{if } \xi > 0 \\ e^{-1} & \text{if } \xi = 0. \end{cases}$$

and

$$\tilde{\delta}_\xi^{b/a*}(p) = \begin{cases} p + \frac{1}{\xi\Delta}\log\left(1 + \frac{\xi\Delta}{k}\right) & \text{if } \xi > 0 \\ p + \frac{1}{k} & \text{if } \xi = 0, \end{cases}$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The functions $H_\xi^{b/a}$ and $\tilde{\delta}_\xi^{b/a*}$

If $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$, then $H_\xi^{b/a}(p) = \frac{A\Delta}{k} C_\xi \exp(-kp)$, with

$$C_\xi = \begin{cases} \left(1 + \frac{\xi\Delta}{k}\right)^{-\frac{k}{\xi\Delta}-1} & \text{if } \xi > 0 \\ e^{-1} & \text{if } \xi = 0. \end{cases}$$

and

$$\tilde{\delta}_\xi^{b/a*}(p) = \begin{cases} p + \frac{1}{\xi\Delta}\log\left(1 + \frac{\xi\Delta}{k}\right) & \text{if } \xi > 0 \\ p + \frac{1}{k} & \text{if } \xi = 0, \end{cases}$$

This corresponds exactly to our framework with entropic costs

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The system of ODEs

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 +$$

$$+\frac{A\Delta}{k} C_\xi \left(1_{q<Q} e^{k\frac{\theta(t,q+\Delta)-\theta(t,q)}{\Delta}} + 1_{q>-Q} e^{k\frac{\theta(t,q-\Delta)-\theta(t,q)}{\Delta}} \right),$$

with final condition $\theta(T,q) = 0$.

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The system of ODEs

$$0 = \partial_t \theta(t,q) - \frac{1}{2}\gamma\sigma^2 q^2 +$$

$$+\frac{A\Delta}{k} C_\xi \left(1_{q<Q} e^{k\frac{\theta(t,q+\Delta)-\theta(t,q)}{\Delta}} + 1_{q>-Q} e^{k\frac{\theta(t,q-\Delta)-\theta(t,q)}{\Delta}}\right),$$

with final condition $\theta(T,q) = 0$.

Change of variables: $v_q(t) = \exp\left(\frac{k\theta(t,q)}{\Delta}\right)$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

A linear system of ODEs

$$v_q'(t) = \alpha q^2 v_q(t) - \eta_\xi \left(1_{q<Q} v_{q+\Delta}(t) + 1_{q>-Q} v_{q-\Delta}(t)\right),$$

with

$$\alpha = \frac{k}{2\Delta}\gamma\sigma^2, \qquad \eta_\xi = AC_\xi$$

and the terminal condition $v(T,q) = 1$.

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

A linear system of ODEs

$$v_q'(t) = \alpha q^2 v_q(t) - \eta_\xi \left(1_{q<Q} v_{q+\Delta}(t) + 1_{q>-Q} v_{q-\Delta}(t)\right),$$

with

$$\alpha = \frac{k}{2\Delta}\gamma\sigma^2, \qquad \eta_\xi = AC_\xi$$

and the terminal condition $v(T, q) = 1$.

This corresponds to

$$B = \begin{pmatrix} -\alpha Q^2 & \eta_\xi & & & \\ \eta_\xi & -\alpha(Q-\Delta)^2 & \eta_\xi & & \\ & \eta_\xi & \ddots & \ddots & \\ & & \ddots & \ddots & \eta_\xi \\ & & & \eta_\xi & -\alpha Q^2 \end{pmatrix}$$

which is symmetric here!

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

Optimal quotes

The optimal quotes in models A ($\xi = \gamma$) and B ($\xi = 0$) are:

$$\delta_t^{b*} = \delta^{b*}(t, q_{t-}) := D_\xi + \frac{1}{k} \ln\left(\frac{v_{q_{t-}}(t)}{v_{q_{t-}+\Delta}(t)}\right)$$

$$\delta_t^{a*} = \delta^{a*}(t, q_{t-}) := D_\xi + \frac{1}{k} \ln\left(\frac{v_{q_{t-}}(t)}{v_{q_{t-}-\Delta}(t)}\right)$$

$$D_\xi = \begin{cases} \frac{1}{\xi\Delta} \log\left(1 + \frac{\xi\Delta}{k}\right) & \text{if } \xi > 0 \\ \frac{1}{k} & \text{if } \xi = 0, \end{cases}$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The optimal quote functions far from T only depend on q:

Asymptotics

$$\delta_\infty^{b*}(q) = \lim_{T \to \infty} \delta^{b*}(0, q) = D_\xi + \frac{1}{k} \ln \left(\frac{f_q^0}{f_{q+\Delta}^0} \right)$$

$$\delta_\infty^{a*}(q) = \lim_{T \to \infty} \delta^{a*}(0, q) = D_\xi + \frac{1}{k} \ln \left(\frac{f_q^0}{f_{q-\Delta}^0} \right)$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The optimal quote functions far from T only depend on q:

Asymptotics

$$\delta_\infty^{b*}(q) = \lim_{T\to\infty} \delta^{b*}(0,q) = D_\xi + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q+\Delta}^0}\right)$$

$$\delta_\infty^{a*}(q) = \lim_{T\to\infty} \delta^{a*}(0,q) = D_\xi + \frac{1}{k}\ln\left(\frac{f_q^0}{f_{q-\Delta}^0}\right)$$

Because B is symmetric, $f^0 \in \mathbb{R}^{2Q/\Delta+1}$ is characterized by a Rayleigh ratio:

$$\underset{\|f\|_2=1}{\operatorname{argmin}} \sum_{|q|\leq Q} \alpha q^2 f_q^2 + \eta_\xi \left(\sum_{q=-Q}^{Q-\Delta} (f_{q+\Delta} - f_q)^2 + (f_Q)^2 + (f_{-Q})^2\right).$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

Continuous counterpart

$\tilde{f}^0 \in L^2(\mathbb{R})$ characterized by:

$$\underset{\|\tilde{f}\|_{L^2(\mathbb{R})}=1}{\operatorname{argmin}} \int_{-\infty}^{\infty} \left(\alpha x^2 \tilde{f}(x)^2 + \eta_\xi \Delta^2 \tilde{f}'(x)^2\right) dx.$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

Continuous counterpart

$\tilde{f}^0 \in L^2(\mathbb{R})$ characterized by:

$$\underset{\|\tilde{f}\|_{L^2(\mathbb{R})}=1}{\operatorname{argmin}} \int_{-\infty}^{\infty} \left(\alpha x^2 \tilde{f}(x)^2 + \eta_\xi \Delta^2 \tilde{f}'(x)^2 \right) dx.$$

$$\tilde{f}^0(x) \propto \exp\left(-\frac{1}{2\Delta} \sqrt{\frac{\alpha}{\eta_\xi}} x^2 \right)$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

Continuous counterpart

$\tilde{f}^0 \in L^2(\mathbb{R})$ characterized by:

$$\underset{\|\tilde{f}\|_{L^2(\mathbb{R})}=1}{\operatorname{argmin}} \int_{-\infty}^{\infty} \left(\alpha x^2 \tilde{f}(x)^2 + \eta_\xi \Delta^2 \tilde{f}'(x)^2\right) dx.$$

$$\tilde{f}^0(x) \propto \exp\left(-\frac{1}{2\Delta}\sqrt{\frac{\alpha}{\eta_\xi}}x^2\right)$$

Hence, we get an approximation of the form:

$$f^0_q \propto \exp\left(-\frac{1}{2\Delta}\sqrt{\frac{\alpha}{\eta_\xi}}q^2\right)$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

Using the continuous counterpart, we get:

Closed-form approximations: optimal quotes (Model A: $\xi = \gamma$)

$$\delta^{b*}_{\infty}(q) \simeq \frac{1}{\Delta\xi}\ln\left(1+\frac{\Delta\xi}{k}\right) + \frac{2q+\Delta}{2}\sqrt{\frac{\gamma\sigma^2}{2kA\Delta}\left(1+\frac{\Delta\xi}{k}\right)^{1+\frac{k}{\Delta\xi}}}$$

$$\delta^{a*}_{\infty}(q) \simeq \frac{1}{\Delta\xi}\ln\left(1+\frac{\Delta\xi}{k}\right) - \frac{2q-\Delta}{2}\sqrt{\frac{\gamma\sigma^2}{2kA\Delta}\left(1+\frac{\Delta\xi}{k}\right)^{1+\frac{k}{\Delta\xi}}}$$

Remark: these formulas are used by many practitioners in Europe and Asia on quote-driven markets.

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

Using the continuous counterpart, we get:

Closed-form approximations: optimal quotes (Model B: $\xi = 0$)

$$\begin{aligned}\delta^{b*}_{\infty}(q) &\simeq \frac{1}{k} + \frac{2q+\Delta}{2}\sqrt{\frac{\gamma\sigma^2 e}{2kA\Delta}}\\ \delta^{a*}_{\infty}(q) &\simeq \frac{1}{k} - \frac{2q-\Delta}{2}\sqrt{\frac{\gamma\sigma^2 e}{2kA\Delta}}\end{aligned}$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

A good way to analyze the result is to consider the spread $\psi = \delta^b + \delta^a$ and the skew $\zeta = \delta^b - \delta^a$.

Closed-form approx.: spread and skew (Model A, $\xi = \gamma$)

$$\psi^*_\infty(q) \simeq \frac{2}{\Delta\xi} \ln\left(1 + \frac{\Delta\xi}{k}\right) + \Delta\sqrt{\frac{\gamma\sigma^2}{2kA\Delta}\left(1 + \frac{\Delta\xi}{k}\right)^{1+\frac{k}{\Delta\xi}}}$$

$$\zeta^*_\infty(q) \simeq 2q\sqrt{\frac{\gamma\sigma^2}{2kA\Delta}\left(1 + \frac{\Delta\xi}{k}\right)^{1+\frac{k}{\Delta\xi}}}$$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

The case $\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta}$

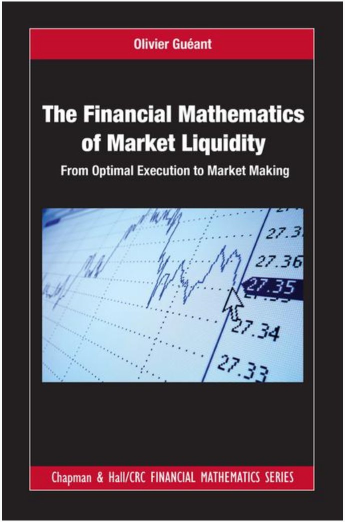
Closed form approx.: spread and skew (Model B, $\xi = 0$)

$$\psi^*_\infty(q) \simeq \frac{2}{k} + \Delta\sqrt{\frac{\gamma\sigma^2 e}{2kA\Delta}}$$

$$\zeta^*_\infty(q) \simeq 2q\sqrt{\frac{\gamma\sigma^2 e}{2kA\Delta}}$$

If you want to know more about market making

If you want to know more about market making



Questions

Thanks for your attention.
Questions.